- A reflection in a plane through $(0,0,0)$ in 3-space
  is given by a matrix
  and the only eigenvectors are:
  - the points on the plane $\leftarrow$ e-value 1
  - the points on the perpendicular $\leftarrow$ e-value -1

- A rotation by some angle θ about an axis through $(0,0,0)$
  is given by a matrix has e-vectors:
  - the non-zero points on the axis $\leftarrow$ e-value 1
  - nothing else (unless θ is a multiple of π)

<u>Remarks</u>: Let A be a matrix.
If $\underline{x}$ is a λ-eigenvector $(A\underline{x} = \lambda\underline{x})$
and if $c \neq 0$ is a number
then $A(c\underline{x}) = cA\underline{x} = c\lambda\underline{x} = \lambda c\underline{x}$
so $c\underline{x}$ is also a λ-eigenvector

- If $\underline{y}$ is another λ-eigenvector
  then $A(\underline{x}+\underline{y}) = A\underline{x} + A\underline{y} = \lambda\underline{x} + \lambda\underline{y} = \lambda(\underline{x}+\underline{y})$
  so $\underline{x}+\underline{y}$ is a λ-eigenvector

<u>Definition</u>: The <u>λ-eigenspace</u> of a matrix A is the set of all λ-eigenvectors and $\underline{0}$
i.e. the set of solutions to the equation

$$A\underline{x} = \lambda\underline{x}$$

i.e. the set of solutions to the equation

$$(A - \lambda I)\underline{x} = \underline{0}$$

Note: If we can write a vector $\underline{x}$ as a sum of eigenvectors, e.g. $\underline{x} = \underline{y} + \underline{z}$ where $\underline{y}$ is a 2-e-vector and $\underline{z}$ is a 3-e-vector

then
$$A\underline{x} = A(\underline{y}+\underline{z}) = A\underline{y} + A\underline{z}$$
$$= 2\underline{y} + 3\underline{z}$$

Finding eigenvectors and eigenvalues

λ is an eigenvalue of an $n \times n$ matrix A

iff there exists $\underline{x} \neq \underline{0}$ such that $A\underline{x} = \lambda \underline{x}$

iff there exists $\underline{x} \neq \underline{0}$ such that $(A - \lambda I)\underline{x} = \underline{0}$

iff the matrix $(A - \lambda I)$ is singular

iff $\det(A - \lambda I) = 0$

So the eigenvalues of A are the zeroes of the function of λ

$$\chi_A(\lambda) = \det(A - \lambda I)$$

("the characteristic polynomial of A")

$\chi_A(\lambda)$ is a polynomial in λ of degree n

$$\chi_A(\lambda) = \pm \lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \dots + c_1\lambda + c_0$$

(c_i are numbers depending on the matrix

<u>Example</u>: $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 4 & 6 \end{pmatrix}$

$$\chi_A(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & 0 \\ 1 & 2-\lambda & -1 \\ 1 & 4 & 6-\lambda \end{pmatrix}$$

$$= -\lambda\left((2-\lambda)(6-\lambda) - (-1)(4)\right)$$

$$= -\lambda(\lambda^2 - 8\lambda + 12 + 4)$$

$$= -\lambda^3 + 8\lambda^2 - 16\lambda$$

solve this:

$$\chi_A(\lambda) = \lambda(-\lambda^2 + 8\lambda - 16)$$

$$= \lambda(4-\lambda)^2$$

so zeroes are 0 and 4

so the eigenvalues of A are 0 and 4.