Example $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 4 & 6 \end{pmatrix}$

$$\begin{aligned} \chi_A(\lambda) &= \det(\lambda I - A) \\ &= \det \begin{pmatrix} \lambda & 0 & 0 \\ -1 & \lambda-2 & 1 \\ -1 & -4 & \lambda-6 \end{pmatrix} \\ &= \lambda\left((\lambda-2)(\lambda-6) - (-4)\right) \\ &= \lambda(\lambda^2 - 8\lambda + 16) \\ &= \lambda(\lambda-4)^2 \end{aligned}$$

so eigenvalues are 0 and 4

Finding eigenvectors:

Once we've found the eigenvalues, we can find the eigenspaces by solving

$(\lambda I - A)\underline{x} = \underline{0}$ for each eigenvalue λ

Example: $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 4 & 6 \end{pmatrix}$ e-values are 0 and 4

0-evectors: solve $(0I - A)\underline{x} = \underline{0}$

i.e. solve $A\underline{x} = \underline{0}$

by Gaussian elimination:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 4 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 4 & 6 \\ 1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 4 & 6 \\ 0 & -2 & -7 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 0 & -8 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{pmatrix} \longleftarrow \begin{pmatrix} 1 & 4 & 6 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{pmatrix}$$

so solutions are $\begin{pmatrix} 8t \\ \frac{-7t}{2} \\ t \end{pmatrix}$ $\left\{\begin{array}{l} z = t \\ x = 8t \\ y = -\frac{7}{2}t \end{array}\right.$

these are the 0-eigenvectors

<u>4-evectors</u>; solve $(4I - A)\underline{x} = 0$

i.e. solve $\begin{pmatrix} 4 & 0 & 0 \\ -1 & 4-2 & 1 \\ -1 & -4 & 4-6 \end{pmatrix} \underline{x} = \underline{0}$

$\Updownarrow$

i.e. $\begin{pmatrix} 4 & 0 & 0 \\ -1 & 2 & 1 \\ -1 & -4 & -2 \end{pmatrix} \underline{x} = 0$

solve it,
get a line of solutions
(one parameter)

<u>Example</u>:

$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad \chi_A(\lambda) = \det(\lambda I - A)$

$= \det \begin{pmatrix} \lambda-2 & -1 & 0 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda-2 \end{pmatrix}$

$= (\lambda - 2)^3$

so only e-value is 2

and the 2-evectors are the non-0 solutions to $(2I - A)\underline{x} = \underline{0}$

$2I - A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad$ solve $\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

solutions are: $= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} t \\ 0 \\ s \end{pmatrix} \quad t \text{ and } s \text{ arbitrary}$$

Diagonalisation:

Definition:

. $n \times n$ matrices A and B are similar if there is an invertible matrix P such that

$$A = P^{-1} B P$$

. An $n \times n$ matrix is diagonalisable if it is similar to a diagonal matrix
i.e. there exists invertible P such that

$$P^{-1} A P \text{ is diagonal}$$

we then say that P diagonalises A.

. what this really means (not on the syllabus):
A and B are similar iff they are ~~a~~ representations of the same map from n-space to itself with respect to (possibly) different co-ordinatisations

P is the "change of co-ordinates" matrix which maps the co-ordinates of a point according to A~~B~~'s co-ordinatisation to the co-ordinates of that point with respect to B~~A~~'s co-ordinatisation.

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[rr, "P"] \arrow[ddd, "A"'] & & \mathbb{R}^n \arrow[ddd, "B"] \\
& n\text{-space} \arrow[ul, "\text{co-ord}_1"'] \arrow[ur, "\text{co-ord}_2"] \arrow[d, "\text{map}"] & \\
& n\text{-space} \arrow[dl, "\text{co-ord}_1"'] \arrow[dr, "\text{co-ord}_2"] & \\
\mathbb{R}^n \arrow[rr, "P"'] & & \mathbb{R}^n
\end{tikzcd}
$$

<u>Remarks</u>: If A and B are similar, say $A = P^{-1}BP$

then : · $B = (P^{-1})^{-1} A P^{-1}$

· <u>$\det(A)$</u> $= \det(P^{-1}BP) = \det(P^{-1})\det(B)\det(P)$

$= \det(P)^{-1}\det(B)\det(P)$

$= \underline{\det(B)}$

· A and B have the same e-values

since if $A\underline{x} = \lambda \underline{x}$, $\underline{x} \neq \underline{0}$

then $P^{-1}BP\underline{x} = \lambda x$

so $BP\underline{x} = P\lambda\underline{x}$

so $B(P\underline{x}) = \lambda(Px)$

and $P\underline{x} \neq \underline{0}$ since P is invertible

so λ is a B e-value

<u>Using diagonalisation to find powers</u>:

If A is diagonalisable, say $P^{-1}AP = D$ is diagonal,

then $A = PDP^{-1}$

so $A^2 = (PDP^{-1})(PDP^{-1}) = PDP^{-1}PDP^{-1}$
$= PD^2P^{-1}$

generally $A^k = PD^kP^{-1}$

finding D^k is easy: just raise the diagonal entries to the kth power

so finding A^k is quite easy!