How to diagonalise

Given an $n \times n$ matrix A, try to find n eigenvectors of A $p_1, \dots, p_n$ such that the matrix

$$P = [p_1\, p_2 \dots p_n]$$

whose columns are $p_1, \dots, p_n$ is invertible.

If we can do this, then $D = P^{-1}AP$ is diagonal and the ith diagonal entry is the e-value of p_i

since
$$P^{-1}AP\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix} = P^{-1}Ap_1 = P^{-1}\lambda_1 p_1 = \lambda_1 P^{-1}p_1 = \lambda_1\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} = \begin{pmatrix}\lambda_1\\0\\\vdots\\0\end{pmatrix}$$

$$\begin{pmatrix}2 & 1\\5 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\5\end{pmatrix}$$

where λ_i is the e-value of p_i

so the first column of $P^{-1}AP$ is $\begin{pmatrix}\lambda_1\\0\\\vdots\\0\end{pmatrix}$

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix}$$

similarly, using $\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}$ etc, the ith column of $P^{-1}AP$ is

$$i\begin{pmatrix}0\\\vdots\\0\\\lambda_i\\0\\\vdots\end{pmatrix}$$

so $P^{-1}AP$ is diagonal with entries $\lambda_1, \dots, \lambda_n$

$$A = \begin{pmatrix}1 & 2\\5 & 4\end{pmatrix}$$

$$\chi_A(\lambda) = \det(\lambda I - A) = \det\begin{pmatrix}\lambda-1 & -2\\-5 & \lambda-4\end{pmatrix} = (\lambda-1)(\lambda-4) - 10 = \lambda^2 - 5\lambda - 6 = (\lambda-6)(\lambda+1)$$

so e-values of A are $6, -1$

solutions of $(6I - A)\underline{x} = 0$

$$6I - A = \begin{pmatrix}5 & -2\\-5 & 2\end{pmatrix}$$

are $(\frac{2t}{5}, t)^T$

so these are the 6-e-vectors of A

e.g. $(2, 5)^T$ is a 6-e-vector $(t = 5)$

the (-1)-e-vectors are $(t, -t)^T$

e.g. $(1, -1)^T$

so let $P = \begin{pmatrix}2 & 1\\5 & -1\end{pmatrix}$

invertible

(if instead we'd used the two (-1)-evectors $\begin{pmatrix}1\\-1\end{pmatrix}$ $\begin{pmatrix}-3\\3\end{pmatrix}$

If no such invertible matrix of e-vectors exists

then A is not diagonalisable

since if A is diagonalisable,

$D = P^{-1}AP$ is diagonal some P invertible

then the columns $p_1, \dots, p_n$ of P are e-vectors for A

since $A p_1 = AP \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$= PD \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$= P \begin{pmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$= \lambda_1 p_1$

it won't work because $\begin{pmatrix} 1 & -3 \\ -1 & 3 \end{pmatrix}$ is not invertible

$P^{-1}AP \begin{pmatrix} 1 \\ 0 \end{pmatrix} = P^{-1}A \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$= P^{-1}\left(6 \begin{pmatrix} 2 \\ 5 \end{pmatrix}\right)$

$= 6 P^{-1} \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$= 6 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$

Si. $P^{-1}AP \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

so $P^{-1}AP = \begin{pmatrix} 6 & 0 \\ 0 & -1 \end{pmatrix}$

<u>Example of undiagonalisablity:</u>

$A = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$

$\chi_A(\lambda) = \begin{pmatrix} \lambda+1 & -1 \\ 1 & \lambda-1 \end{pmatrix}$

$= (\lambda+1)(\lambda-1) + 1$

$= \lambda^2$

only e-value is 0

solutions to $A\underline{x} = \underline{0}$ are $\begin{pmatrix} t \\ t \end{pmatrix}$

so P would have to be $\begin{pmatrix} t & s \\ t & s \end{pmatrix}$

which is not invertible.

so $\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$ is not diagonalisable

<u>Finding an invertible matrix of e-vectors</u>:

If $P = [p_1\, p_2 \ldots p_n]$ is invertible, then at least we have that no two columns are multiples of each other ($p_i = c p_j$ some c)

In simple examples this condition is sufficient for invertibility.

Generally, we'll see later that P is invertible iff the columns $p_1, \ldots, p_n$ "span" $\mathbb{R}^n$

i.e every vector can be written as a sum of multiples of the p_i

i.e. for any $\underline{x}$

$$\underline{x} = c_1 p_1 + \ldots + c_n p_n$$

i.e. A is diagonalisable iff the e-vectors of A span the whole space

<u>Remark</u>; If A is diagonalisable, $D = P^{-1}AP$ is diagonal

then A and D have the same e-values (since similar)

so P must have as a column an e-vector of each e-value.

It might have more than one of a given e-value.

Fact: If ~~ker~~ $p_1, \dots, p_n$ have different e-values then $P = [p_1, \dots p_n]$ is invertible

Theorem: If an $n \times n$ matrix A has n distinct e-values i.e. if $\chi_A(\lambda)$ splits into distinct linear factors, then A is diagonalizable