//(real evalue: 0.57; evector: (0.44, 0.56, 0.70)^T)

What happens if we introduce a 4th species, and change the matrix to

$$\begin{pmatrix} 1.5 & -0.5 & 0 & 0 \\ 0 & 1.5 & -0.5 & 0 \\ 0 & 0 & 1.5 & -0.5 \\ \mathbf{-0.5} & \mathbf{0} & \mathbf{0} & \mathbf{1.5} \end{pmatrix}$$

real epairs: **(1,(1,1,1,1))**; **(2,(1,-1,1,-1))**

Remark: these matrices are not diagonalisable over the reals (although they are over the complex numbers! See later)

Example - network analysis and centrality:

Suppose we have a network consisting of some nodes and links from nodes to other nodes.

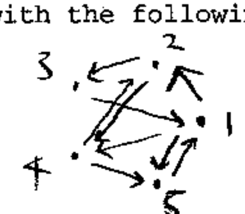
e.g. webpages and weblinks
journals and citations
"tweeters" and "following"

We want to determine which nodes are most "important" in the network.

Idea: an important node is one which is linked to from important nodes.

This seems circular! It is, but it can make sense anyway...

Say we have five webpages, with the following link graph:



Take a large number (m of people. Put each in front of a computer with a browser loaded to a random one of our five pages. Suppose that they're really just browsing aimlessly, and they click links at random.

Question: where will they end up?

Definition:

A column vector is _stochastic_ if each entry is non-negative and the sum of the entries is 1.

Idea: we can interpret a stochastic vector as giving the probability of being in each of **N** states, when we know we have to be in one of them.

A matrix is _stochastic_ if all its columns are.

Idea: the jth column represents the probabilities of something currently in the jth state changing to each of the states in the next step.

Fact [Perron-Frobenius for stochastic matrices] (not on syllabus):

If **A** is a stochastic square matrix and for some **n**, all entries of **A^n** are

Let $\underline{x}_t$ be the 5-vector whose entries are the number of people (in millions) at the various pages after following t links

so $\underline{x}_0 = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{pmatrix}$ $\quad \underline{x}_1 = \begin{pmatrix} 0.4 \\ \frac{1}{3}0.2 + \frac{1}{2}0.2 \\ 0.1 \\ \\ \\ \end{pmatrix}$

$$\underline{x}_1 = A\underline{x}_0$$

where $A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$

$$\underline{x}_2 = A\underline{x}_1 = A^2 \underline{x}_0$$

$$\underline{x}_t = A^t \underline{x}_0$$

turns out that $\underline{v}_t = \begin{pmatrix} 0.3 \\ 0.2 \\ 0.1 \\ 0.2 \\ 0.2 \end{pmatrix}$ is "stable"

i.e. $A\underline{v}_t = \underline{v}_t$

i.e. $\underline{v}_t$ is a 1-e-vector

and whatever $\underline{x}_0$ is,

$$\underline{x}_t \xrightarrow[t \to \infty]{} \underline{v}_t$$

positive, then it has a unique stochastic eigenvector **v_1**, which has **e**-value 1, and for any stochastic **x**

    lim_{n -> \inf} A^n x = v_1

$$\lim_{n\to\infty} A^n x = v_1$$

Stochastic diagonalisable example:

    (0   1   2/3)
    (1/3   0   1/3)
    (2/3   0   0)

$$A\underline{x} = \underline{x}$$

$$\begin{pmatrix} 0 & 1 & 2/3 \\ 1/3 & 0 & 1/3 \\ 2/3 & 0 & 0 \end{pmatrix}$$

$(I - A)\underline{x} = 0$ (e.g. blinking links!)

evalues and evectors are:

    1, -2/3, -1/3

$(-\tfrac{2}{3} I - A)\underline{x} = 0$ with corresponding evectors

$(9, 5, 6)^T$ evalue 1

$(1, 0, -1)^T$ evalue ~~1/6~~ $-2/3$

$(1, 1, -2)^T$ evalue ~~-1/12~~ $-1/3$

e-value $1, -\tfrac{2}{3}, -\tfrac{1}{3}$

So stochastic 1-e-vector is **v_1 = (9/20, 5/20, 6/20)^T**

So this is the limit behaviour.

$\begin{pmatrix} 9 \\ 5 \\ 6 \end{pmatrix}$ 1-e-vect

Any stochastic **e**-vector **v** can be written as

    v = v_1 + x_2 + x_3

where **x_2** is a ~~1/6~~ $-\tfrac{2}{3}$-evector and **x_3** is a ~~-1/12~~ $-\tfrac{1}{3}$-evector.

e.g. **(1/3,1/3,1/3) = v_1 + (1/12, 1/12, -1/6)^T + (1/5, 0, -1/5)^T**

So **A^n v = ...**

$$v_1 = \begin{pmatrix} 9/20 \\ 5/20 \\ 6/20 \end{pmatrix}$$

Another:

    (1/4 1/3 1/3)   (2/3   0   -3/2)
    (1/4 1/3 1/6)   P= (1/2 -1   1/2)   D=diag(1,1/6,-1/12)
    (1/2 1/3 1/2)   (1   1   1)

evalues and evectors are:

    1, 1/6, -1/12

with corresponding evectors

(2/3, 1/2, 1) evalue 1

(0, -1 , 1) evalue **1/6**

(-3/2, 1/2, 1) evalue **-1/12**

So stochastic 1-e-vector is (**4/13 3/13 6/13)^T**

So this is the limit behaviour.

$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $-\tfrac{2}{3}$-e-vect

$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ $-\tfrac{1}{3}$-e-vect

1-e vector

$-\tfrac{1}{3}$-evector

$-\tfrac{2}{3}$ e-vector

$$\begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} = v_1 + \begin{pmatrix} 1/12 \\ 1/12 \\ -1/6 \end{pmatrix} + \begin{pmatrix} 1/5 \\ 0 \\ -1/5 \end{pmatrix}$$

$$A^n \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} = A^n v_1 + A^n \begin{pmatrix} 1/12 \\ 1/12 \\ -1/6 \end{pmatrix} + A^n \begin{pmatrix} 1/5 \\ 0 \\ -1/5 \end{pmatrix}$$

$$= v_1 + \left(-\tfrac{1}{3}\right)^n \begin{pmatrix} 1/12 \\ 1/12 \\ -1/6 \end{pmatrix} + \left(-\tfrac{2}{3}\right)^n \begin{pmatrix} 1/5 \\ 0 \\ -1/5 \end{pmatrix}$$