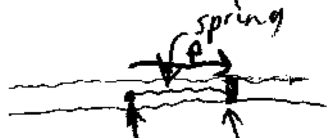
Simple harmonic motion



fixed point

m a 1kg mass constrained to the line attached to the spring

At time t, let $p(t)$ be the position of the mass on the line
let $q(t)$ be its velocity

Say the spring is such that the force on the mass at time t is $-p(t)$

So we have the following differential equations:

$$\frac{dp}{dt} = q \qquad \frac{dq}{dt} = -p \qquad \begin{pmatrix} F = ma \\ m=1 \end{pmatrix}$$

"correct" thing to do is to apply the theory of differential equations.

But lets consider trying to find a discrete approximation.

Let $\Delta t > 0$ be small, say $\Delta t = 0.01$

let $x_n = \begin{pmatrix} p(n\Delta t) \\ q(n\Delta t) \end{pmatrix} = \begin{pmatrix} p_n \\ q_n \end{pmatrix}$

then $p(n\Delta t + \Delta t)$ is roughly $p(n\Delta t) + \Delta t\, q(n\Delta t)$
$q(n\Delta t + \Delta t)$ is roughly $q(n\Delta t) - \Delta t\, p(n\Delta t)$

so (roughly) $x_{n+1} = \begin{pmatrix} p_n + \Delta t\, q_n \\ q_n - \Delta t\, p_n \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{pmatrix}}_{A} x_n$

Now $$\chi_A(\lambda) = \det\begin{pmatrix} \lambda-1 & -\Delta t \\ \Delta t & \lambda-1 \end{pmatrix} = (\lambda-1)^2 + (\Delta t)^2 = \lambda^2 - 2\lambda + 1 + (\Delta t)^2$$

use quadratic formula to find zeroes:

$$\frac{2 \pm \sqrt{4-4(1+(\Delta t)^2)}}{2} = 1 \pm \frac{\sqrt{-4(\Delta t)^2}}{2} = 1 \pm \sqrt{-(\Delta t)^2}$$

no (real) roots

so no (real) e-vectors or e-values

But ...

<u>Complex numbers</u>

Basic idea: we declare that we can take square roots of negative numbers, try to make sense of it.

Introduce a new symbol, i, to be a square root of -1. So we declare that

$$\boxed{i^2 = -1}$$

Now we extend the real numbers with their usual addition and multiplication to include i such that the nice properties still hold;

$0 + z = z$

$z + w = w + z$

$1z = z$

$zw = wz$

$z(w+u) = zw + zu$

$(z+w)+u = z+(w+u)$

$(zw)u = z(wu)$

Now i is one of our numbers and any real number is,
so bi must be for any real b
and so so must $a+bi$ for any real a and b.
We'll see that we can stop there

so we define a <u>complex number</u> to be a number of the form $a+bi$ for a and b real

$a+bi = c+di$ iff $a=c$ and $b=d$

$\mathrm{Re}(a+bi)=a$ "real part"

$\mathrm{Im}(a+bi)=b$ "imaginary part"

an <u>imaginary</u> number is a complex number z with $\mathrm{Re}(z)=0$
e.g. i, $5i$, $-\pi i$

Defining addition and multiplication:
if we want the laws above to hold, we have no choice!

$$
\begin{aligned}
(a+bi)+(c+di) &= a+bi+c+di \\
&= a+c+bi+di \\
&= (a+c)+(b+d)i
\end{aligned}
$$

so $\boxed{(a+bi)+(c+di) = (a+c)+(b+d)i}$

$$
\begin{aligned}
(a+bi)(c+di) &= (a+bi)c + (a+bi)di \\
&= ac+bci + adi + bdi^2
\end{aligned}
$$

$$= ac + bci + adi - bd \qquad (i^2 = -1)$$

$$= (ac - bd) + (ad + bc)i$$

so $$\boxed{(a+bi)(c+di) = (ac-bd) + (ad+bc)i}$$

Then it turns out that the laws of addition and multiplication given above do hold