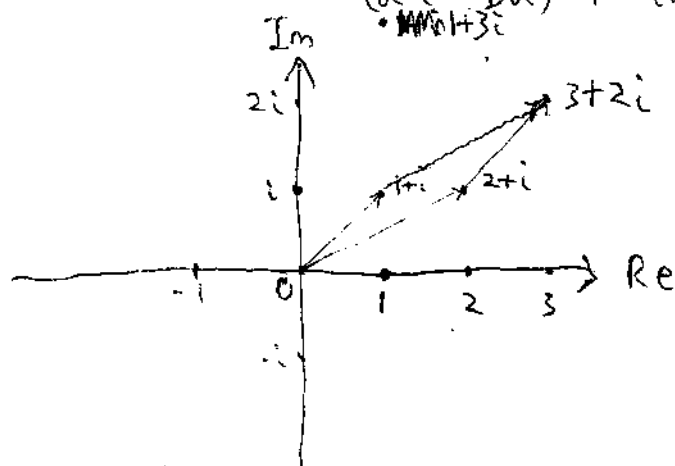


$$a + bi$$

$$i^2 = -1$$

$$(a+bi)(c+di) = ac + adi + bci + bdi^2$$

$$= (ac - bd) + (ad + bc)i$$



addition of complex numbers
vector addition

"Argand diagram"
"Complex plane"

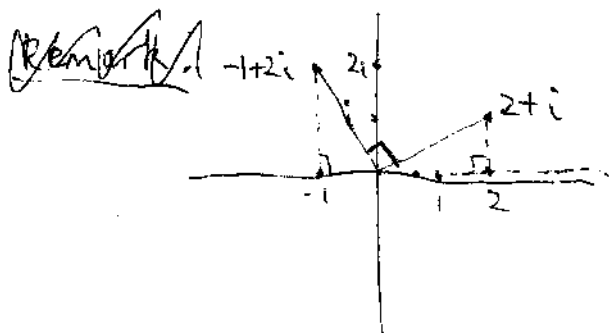
$$(1+i)(2+i) = 2 + i + 2i + i^2$$

$$= 1 + 3i$$

$$i^2 = -1 \quad i^3 = i^2 i = -i \quad i^4 = i^2 i^2 = (-1)^2 = 1$$

$$(-i)^2 = (-1)^2 i^2 = i^2 = -1$$

$$\text{(generally)} \quad (-z)^2 = (-1)z^2 = (-1)^2 z^2 = z^2$$



$$i(2+i) = -1+2i$$

Remark: For any complex $z = a + bi$,
 $zi = i z = -b + ai$
 so iz is what you get if you rotate
 z by $\pi/2$ counterclockwise around 0

Polar form

Definition: For θ a real number,

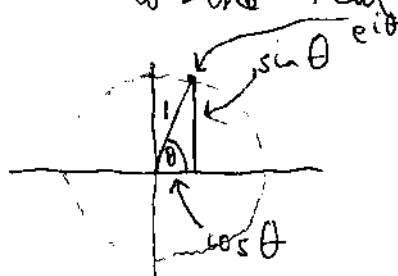
$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$\cos \theta = \operatorname{Re}(e^{i\theta})$$

$$\sin \theta = \operatorname{Im}(e^{i\theta})$$

= [the point on the unit circle
 at angle θ counterclockwise
 to the real axis]

(also $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$
 $\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$
 think about it)



~~Proof~~

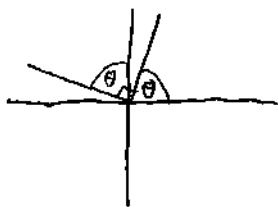
$$\sqrt{(\cos \theta)^2 + (\sin \theta)^2}$$

$$= \sqrt{1} = 1$$

Lemma: For any real θ and ϕ ,

$$e^{i\theta} \cdot e^{i\phi} = e^{i(\theta+\phi)}$$

Proof: If $\phi = 0$, so $e^{i\phi} = e^0 = 1$, then $e^{i\theta} \cdot e^{i\phi} = e^{i\theta} = e^{i(\theta+0)}$
 If $\phi = \pi/2$, so $e^{i\phi} = i$, then $e^{i\theta} \cdot e^{i\phi} = e^{i\theta} i = [e^{i\theta} \text{ rotated by } \pi/2]$
 $= e^{i(\theta+\pi/2)}$



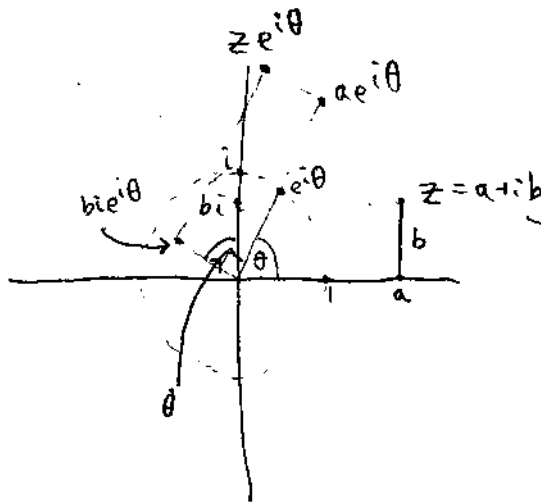
So multiplying $e^{i\theta}$ with 1 and with i
acts as rotation by θ

so if $z = a + bi$

$$\begin{aligned} \text{then } e^{i\theta} z &= e^{i\theta} (a + bi) = e^{i\theta} a + e^{i\theta} bi \\ &= [a \text{ rotated by } \theta] \\ &\quad + [bi \text{ rotated by } \theta] \end{aligned}$$

$$\begin{aligned} &= [a + bi \text{ rotated by } \theta] \\ &= [z \text{ rotated by } \theta] \end{aligned}$$

so $e^{i\theta} e^{i\phi} = [e^{i\phi} \text{ rotated by } \theta] = e^{i(\theta+\phi)}$



A diagram of the complex plane showing the complex number $\frac{2+i}{\sqrt{5}} = e^{i\theta}$ in the first quadrant. The angle θ is indicated between the vector and the positive real axis. Below the diagram, the following calculation is shown:

$$\begin{aligned} \left(\frac{2+i}{\sqrt{5}} \right)^2 &= \frac{4 - 1 + 4i}{5} \\ &= \frac{3 + 4i}{5} \\ &= e^{i2\theta} \end{aligned}$$