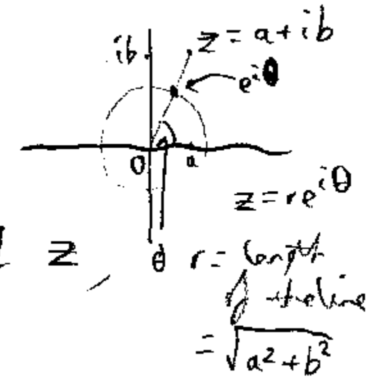
Alternative proof:

$$e^{i\theta} \cdot e^{i\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$$
$$= \cos\theta\cos\phi - \sin\theta\sin\phi + i(\cos\theta\sin\phi + \sin\theta\cos\phi)$$
$$= \cos(\theta+\phi) + i\sin(\theta+\phi)$$
$$= e^{i(\theta+\phi)}$$

Any complex number z can be written as $re^{i\theta}$ for some real r and some real θ



This is the <u>polar form</u> of z

r is the <u>absolute value</u> or <u>modulus</u> of z, written $|z|$

θ is <u>an argument</u> for z written $\arg(z)$

If $z = a+bi$
then $r = \sqrt{a^2+b^2} = |z|$
$\theta = \arctan\left(\frac{b}{a}\right)$

Note $e^{2\pi i} = 1$ and $e^{2\pi i k} = 1$ for any integer k

So for any z, $ze^{2\pi i k} = z$

so $re^{i\theta} = re^{i(\theta+2\pi k)}$

so the argument is unique only up to adding multiples of 2π

Multiplying in polar form:

$$(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i\theta_1} e^{i\theta_2}$$
$$= (r_1 r_2) e^{i(\theta_1+\theta_2)}$$

Division:

If $z \neq 0$, say $z = re^{i\theta}$, then

$$(r^{-1}e^{-i\theta})(re^{i\theta}) = r^{-1}r\, e^{-i\theta+i\theta}$$
$$= 1 \cdot 1$$
$$= 1$$

$$(re^{i\theta+2\pi i})^{-1} = r^{-1}e^{-i\theta-2\pi i} = r^{-1}e^{-i\theta}$$

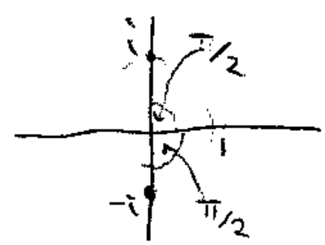
so $r^{-1}e^{-i\theta}$ is the reciprocal of $z = re^{i\theta}$

we write it as z^{-1} or $1/z$

So now for any complex w,

let w/z be wz^{-1}

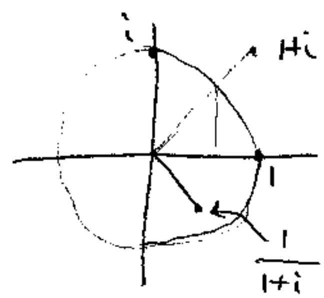
so $(w/z)\, z = w$.



Examples

- $\frac{1}{i} = \frac{1}{e^{i\pi/2}} = e^{-i\pi/2} = -i$

  $i \cdot (-i) = -i^2 = -(-1) = 1$



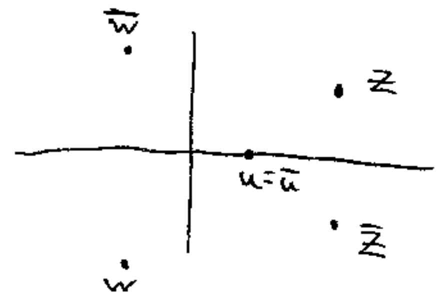
- $$\frac{1}{1+i} = \frac{1}{\sqrt{2}\, e^{i\pi/4}}$$
  $$= \frac{1}{\sqrt{2}} e^{-i\pi/4}$$
  $$= \frac{1}{\sqrt{2}}\left(\cos(-\pi/4) + i\sin(-\pi/4)\right)$$
  $$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \frac{1}{2} - \frac{i}{2} = \frac{1-i}{2}$$

$$\left(\frac{1-i}{2}\right)(1+i) = \frac{1-i^2}{2} = \frac{2}{2} = 1$$

<u>Definition</u>: The <u>complex conjugate</u> of $z = a+ib$

is $\bar{z} = a - ib$ ("reflect in the real axis")

in polar form, $\overline{re^{i\theta}} = re^{-i\theta}$



<u>Remark</u>: $\underline{z\bar{z}} = \overset{z}{re^{i\theta}}\, re^{-i\theta} = r^2 e^{i\theta - i\theta} = r^2 = \underline{|z|^2}$

so $1/z = \bar{z}/|z|^2$

so $w/z = \dfrac{w\bar{z}}{|z|^2}$