Taking n^{th} roots:

The solutions to $z^n = 1$

are precisely $z = e^{(k 2\pi i/n)}$ k an integer

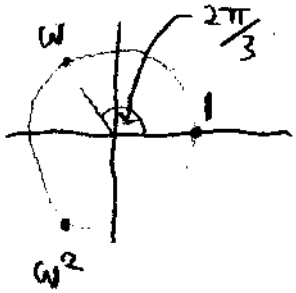
there are n of these:

$$1 = e^0,\ e^{2\pi i/n},\ e^{2(2\pi i)/n},\ e^{3(2\pi i)/n},\ \dots,\ e^{(n-1)(2\pi i)/n}$$

$e^{n \cdot 2\pi i/n} = e^{2\pi i} = 1$

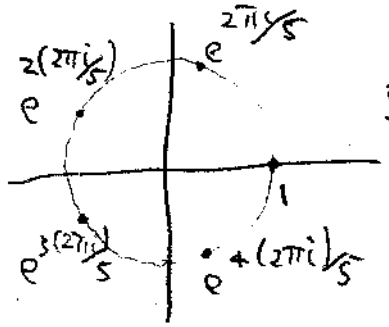
e.g. cube roots of 1 are 1, $e^{2\pi i/3}$, $e^{2(2\pi i)/3}$

ω ω^2



$$\omega = e^{2\pi i/3} = \cos\tfrac{2\pi}{3} + i\sin\tfrac{2\pi}{3}$$
$$= -\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}$$

$$\omega^2 = e^{2(2\pi i/3)} = e^{-2\pi i/3} = \overline{e^{2\pi i/3}} = \overline{\omega} = -\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}$$



5th roots of 1

Now consider $z^n = \alpha$ α a complex number

If $\beta^n = \alpha$ is a solution, then also $\left(\beta e^{k 2\pi i/n}\right)^n = \beta^n \cdot 1 = \alpha$

so we get n solutions (assuming $\alpha \neq 0$)

Concretely: ← polar form of α

the solutions to $z^n = \alpha = re^{i\theta}$

are $r^{1/n} e^{i\theta/n + k2\pi i/n}$ k an integer

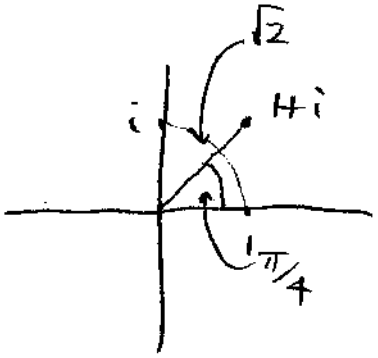
↑ positive real number

$\left(r^{1/n} e^{i\theta/n}\right)^n = re^{i\theta}$

n solutions

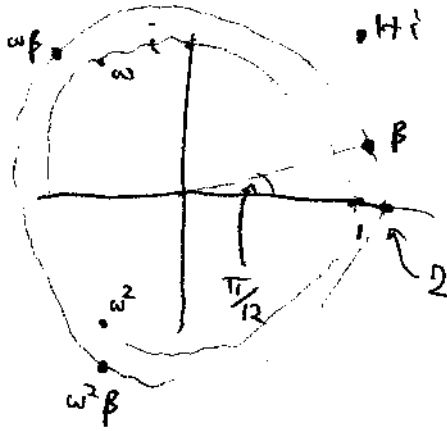
e.g. cube roots of $1+i$

$1+i = \sqrt{2}\, e^{i\pi/4}$



so the cube roots are

$(\sqrt{2})^{1/3} e^{i\pi/12 + k2\pi i/3}$



$\beta^3 = (\omega\beta)^3 = (\omega^2\beta)^3 = 1+i$

Not on syllabus

Fact: "Fundamental Theorem of Algebra"

A polynomial of degree d with complex coefficients

$f(z) = \alpha_d z^d + \alpha_{d-1} z^{d-1} + \dots + \alpha_0$ $\quad \alpha_0, \dots, \alpha_d$ complex numbers

splits into d linear with complex coefficients

$f(z) = \alpha_d (z - \beta_1)(z - \beta_2) \dots (z - \beta_d)$ $\quad \beta_1, \dots, \beta_d$ complex

so $\beta_1, \dots, \beta_d$ are the zeroes of f

e.g. any characteristic polynomial splits into linear factors

so any $d \times d$ matrix does have e-values

$\beta_1, \dots, \beta_d$ complex numbers

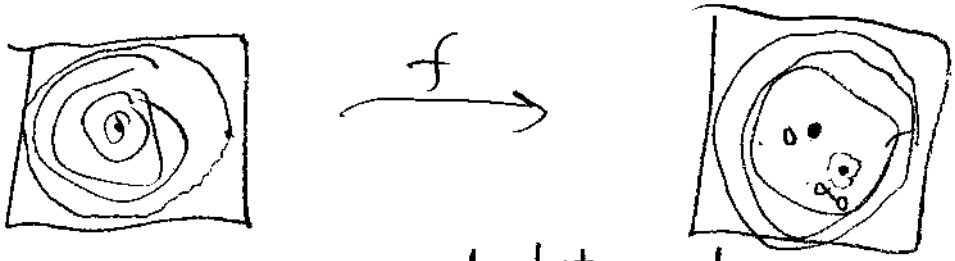
(not necessarily distinct

e.g. $(z-i)^2$)

Idea of proof of fact:

Enough to see that f has a zero.

Suppose it doesn't.



for z of big enough absolute value:

$f(z) \sim \alpha_1 z^d$