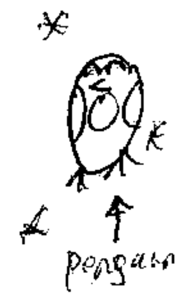
The fundamental property

For any u in $\mathbb{R}^3$

$$u\cdot(v\times w) = \det\begin{pmatrix} | & | & | \\ u & v & w \\ | & | & | \end{pmatrix}$$

Proof: cofactor expansion down first column

$$\det\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix} = u_1 \begin{vmatrix} v_2 & w_2 \\ v_3 & w_3 \end{vmatrix}$$



$$u(v\times w) = \begin{matrix} -u_2 \\ +u_3 \end{matrix} \left| \quad \right|$$

So if v and w are collinear then $v\times w = 0$
and for any v, w
$$v\cdot(v\times w) = 0 = w\cdot(v\times w)$$
So $v\times w$ is normal to v and to w

Now suppose v, w not collinear

Fact: For any u, v, w in $\mathbb{R}^3$

$$\det\begin{pmatrix} | & | & | \\ u & v & w \\ | & | & | \end{pmatrix} = \text{SignedVol}(\text{the parallelepiped of } u, v, w)$$

AKA a 3-D rhombus

$$u\cdot(v\times w) = \det\begin{pmatrix} | & | & | \\ u & v & w \\ | & | & | \end{pmatrix}$$