2A03 Ass 2

§4.1 2. $f_x = 0$

so $f_{xx} = f_{xy} = 0$

$f_y > 0$

$f_{yy} > 0$

$f_{yx} = 0$

6. $z = x^y + (\ln y)^x$

$z_x = yx^{y-1} + (\ln(\ln y))(\ln y)^x$

$z_{xx} = y(y-1)x^{y-2} + (\ln(\ln(y)))^2(\ln y)^x$

$z_{xy} = x^{y-1} + y(\ln x)x^{y-1} + \frac{1}{y \ln y}(\ln y)^x + \frac{(\ln(\ln y))x(\ln y)^{x-1}}{y}$

$z_y = (\ln x)x^y + \frac{x(\ln y)^{x-1}}{y}$

$z_{yx} = \frac{x^y}{x} + (\ln x)yx^{y-1} + \frac{(\ln y)^{x-1}}{y} + \frac{x(\ln(\ln y))(\ln y)^{x-1}}{y}$

$(= z_{xy})$

14.(a) $u = e^{ax+bt}$ $u_t = be^{ax+bt}$

$u_{xx} = a^2 e^{ax+bt}$

so $u_t = \frac{b}{a^2} u_{xx}$

(b) $u = t^{-1/2} e^{-x^2/t}$ $u_t = \left(-\frac{1}{2}t^{-3/2} + t^{-1/2}\left(\frac{x^2}{t^2}\right)\right)e^{-x^2/t}$

$u_x = -t^{-1/2}\frac{(2x)}{t} e^{-x^2/t}$

$u_{xx} = -t^{-1/2}\left(\frac{2}{t}\right)\left(1 + x\frac{-2x}{t}\right)e^{-x^2/t}$

$= (-2t^{-3/2} + 4x^2t^{-5/2})e^{-x^2/t}$

so $u_t = \frac{1}{4}u_{xx}$

§4.2 12 $0.087\ln(1.087) = t\ln(1+t)$ with $t = 0.087$

2nd order Taylor for $t\ln(1+t)$ at 0;

$t\ln(1+t) = 0\ln 1 + t\frac{d}{dt}(t\ln(1+t))\Big|_{t=0} + \frac{1}{2}t^2\frac{d^2}{dt^2}(t\ln(1+t))\Big|_{t=0} + \text{rem}^n$

$= 0 + (0 + \ln 1)t + \frac{1}{2}\left(\frac{2+0}{(1+0)^2}\right)t^2 + R_2$

$= 0 + 0 + t^2 + R_2 = t^2 + R_2$

$\frac{d}{dt}(t\ln(1+t)) = \frac{t}{1+t} + \ln(1+t)$

$\frac{d^2}{dt^2}(t\ln(1+t)) = \frac{1}{1+t} - \frac{t}{(1+t)^2} + \frac{1}{1+t} = \frac{2+t}{(1+t)^2}$

So 1st order approx is 0

which isn't much use

2nd order is $(0.087)^2 = 7.57 \times 10^{-3}$

which isn't bad - in fact, $0.087\ln(1+0.087)$

$= 7.26 \times 10^{-3}$

32. $f(x,y) = ye^{x^2}$

$\nabla f(x,y) = (2xye^{x^2}, e^{x^2})$

$Hf(x,y) = \begin{pmatrix} 2y(1+2x^2)e^{x^2} & 2xe^{x^2} \\ 2xe^{x^2} & 0 \end{pmatrix}$

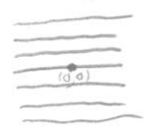
Taylor at (0,0);

$f(x,y) = f(0,0) + \nabla f(0,0)\cdot\underline{x} + \frac{1}{2}\underline{x}^t Hf(0,0)\underline{x} + R_2$

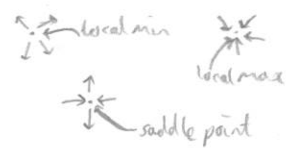
$= 0 + 0x + 1y + \frac{1}{2}(0x^2 + 0xy + 0y^2) + R_2$

$= y + R_2$

so near (0,0) contour diagram looks (at least roughly) like that of y:



§4.3 6.



28. $f(x,y) = 2x^2 + y^2 + 2$ on $D = \{(x,y) \mid x^2+y^2 \leq 2^2\}$

critical points in interior;

$\nabla f = (4x, 2y) = \underline{0} \iff (x,y) = (0,0)$

$Hf = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$ $|Hf(\underline{0})| = 8 > 0$, so local extremum

$f_{xx}(\underline{0}) = 4 > 0$, so $(0,0)$ is a local min.

$f(0,0) = 2$.

Boundary: $\partial D = \{(x,y) \mid x^2+y^2 = 2^2\}$

on ∂D, $f(x,y) = x^2 + (x^2+y^2) + 2$

$= x^2 + 4 + 2$

$= x^2 + 6$

so min when $x = 0$; $f(0,2) = f(0,-2) = 6$

but $6 > 2$

So global min is 2, achieved at (0,0).

28 cont^d: on ∂D, $f(x,y) = x^2 + 6$ is max at $x = \pm 2$

$f(2,0) = f(-2,0) = 10$

since no local max in interior, this must be global max.

So global max is 10, achieved at (-2,0) and at (2,0).

§4.4 6.

local extrema where grad perpendicular to curve (as with Lagrange multipliers); field on curve coming in towards point ⇒ max

going out ⇒ min.

22 minimise $T(x,y) = 2x^2 - xy + 2y^2 + 10$

subject to $g(x,y) = x^2 + y^2 = 50$

Lagrange: local extrema are sol^{ns} to

$\nabla T = \lambda \nabla g$

i.e. $(4x - y, -x + 4y) = \lambda(2x, 2y)$

$\lambda 2x = 4x - y \Rightarrow y = (4 - 2\lambda)x$

$2\lambda y = -x + 4y \Rightarrow x = (4 - 2\lambda)y$

so $y = (4 - 2\lambda)^2 y$

so $(4 - 2\lambda)^2 = 1$

so $\lambda = 3/2$ or $5/2$

and $y = x$ or $-x$ respectively

Now $x^2 + y^2 = 50$

so $y = \pm x \Rightarrow (x,y) = \left(\pm\sqrt{\frac{50}{2}}, \pm\sqrt{\frac{50}{2}}\right)$

$= (\pm 5, \pm 5)$

(4 possibilities)

so min is at one of these.

$\boxed{T(5,5) = T(-5,-5) = 85}$ ← global min

$T(5,-5) = T(-5,5) = 135$ ← global max