2A03 Ass 4

§5.3 10. 

$F(x,y) = (e^{x+y}, e^{x-y})$

$c_1(t) = (0,t)$ $\quad c_1'(t) = (0,1)$

$c_2(t) = (t, 1-t)$ $\quad c_2'(t) = (1,-1)$

$c_3(t) = (1-t, 0)$ $\quad c_3'(t) = (-1,0)$

$$\int_c F\cdot ds = \sum_i \int_{c_i} F\cdot ds = \int_0^1 (e^{0+t}, e^{0-t})\cdot(0,1)\,dt + \int_0^1 (e^1, e^{2t-1})\cdot(1,-1)\,dt + \int_0^1 (e^{1-t}, e^{1-t})\cdot(-1,0)\,dt$$

$$= \int_0^1 (e^{-t} + e - e^{2t-1} - e^{1-t})\,dt$$

$$= -(e^{-1}-1) + e - \tfrac{1}{2}(e - e^{-1}) + (e^0 - e)$$

$$= 1 - \tfrac{1}{2}e^{-1} - \tfrac{1}{2}e + 1$$

$$= \underline{2 - \tfrac{1}{2}(e + \tfrac{1}{e})}$$

16. $F(x) = (a,b) = a$ constant $\quad c(\theta) = (\cos\theta, \sin\theta)$

Work done $= \int_c F\cdot ds = \int_0^{2\pi} a\cdot(-\sin\theta, \cos\theta)\,d\theta$

$$= \int_0^{2\pi} (b\cos\theta - a\sin\theta)\,d\theta$$

$$= b\sin\theta\Big|_0^{2\pi} + a\cos\theta\Big|_0^{2\pi}$$

$$= 0 - 0 = \underline{0}$$

(or could note that $F = \nabla f$ where $f(x) = ax + by$ so any circulation is 0)

§5.4 20. $\operatorname{dom} F = \mathbb{R}^3$

$\operatorname{curl} F = (x - x,\ e^x\cos z - e^x \cos z,\ z - z) = 0$

so must be a gradient vector field ($\mathbb{R}^3$ is simply connected)

Indeed, let $f_0(x) = xyz + e^x\sin z + y^3/3 - e^y$

then $\nabla f_0(x) = F(x)$

so $F = \nabla f$ iff $f(x) = f_0(x) + c$

28. $\operatorname{curl} F = (0-0,\ 0-0,\ x^3 - 0) = (0,0,x^3)$

not constantly 0

so F not a gradient vector field.

T/F: 1. Yes, since path integral of ∇f = diff between values of f at end points

$\| \quad \int \nabla f\cdot c'\,dt = 0$ ← since $c' \perp \nabla f$

(oh, I didn't set this one. Ignore me!)

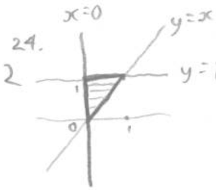
2. Yes: $F(x) = (a,b) = \nabla(ax+by)$

6. No; $\int_c 2\,ds$ = twice circumference $= 2(2\pi) = 4\pi$

12. Yes; $\int_{c_1}(3F - 2G)\cdot ds = 3\int_{c_1} F\cdot ds - 2\int_{c_1} G\cdot ds$

$$= 3\int_{c_2} F\cdot ds - 2\int_{c_2} G\cdot ds$$

$$= \int_{c_2} (3F - 2G)\cdot ds$$

§6.2 24. 

$D = \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le y\}$

$$\iint_D \ln(xy)\,dA = \int_0^1\int_0^y \ln(xy)\,dx\,dy$$

$$= \int_0^1 \frac{1}{y}\left[xy(\ln(xy) - 1)\right]_0^y dy \quad \text{(see ①)}$$

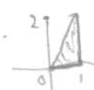
$$= \int_0^1 y(\ln(y^2) - 1)\,dy \quad \text{(see ②)}$$

$$= \tfrac{1}{2}\int_0^1 \ln u\,du - 1 \quad (u = y^2)$$

$$= \tfrac{1}{2}\left[u(\ln u - 1)\right]_0^1 - 1$$

$$= \tfrac{1}{2}((\ln 1 - 1) - 0) - 1$$

$$= \underline{-\tfrac{3}{2}}$$

30. 

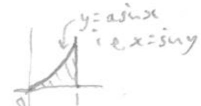
$$\int_0^1\int_0^{2x} (9 - x^2 - y^2)\,dy\,dx$$

$$= \int_0^1 \left(2x(9-x^2) - \frac{(2x)^3}{3}\right)dx$$

$$= \int_0^1 \left(18x - \frac{14}{3}x^3\right)dx$$

$$= 9 - \frac{14}{12} = \underline{9 - \tfrac{7}{6}}$$

§6.3 22. $\displaystyle\int_0^1\int_0^{\operatorname{asin} x} y^2\,dy\,dx = \iint_D y^2\,dA$



$$= \int_0^{\operatorname{asin} 1}\int_{\sin y}^1 y^2\,dx\,dy$$

$$= \int_0^{\pi/2} (1 - \sin y)y^2\,dy$$

$$= \pi/2 - \int_0^{\pi/2} y^2\sin y\,dy$$

$$= \frac{\pi}{2} - \left[(2-y^2)\cos y + 2y\sin y\right]_0^{\pi/2}$$

$$= \frac{\pi}{2} - (\pi - 2)$$

$$= 2 - \pi/2$$

$$\int y^2\sin y\,dy = -y^2\cos y + \int 2y\cos y\,dy = -y^2\cos y + 2\left(y\sin y - \int\sin y\,dy\right) = (2-y^2)\cos y + 2y\sin y$$

① $\int \ln x\,dx = \int 1\cdot\ln x\,dx = x\ln x - \int \frac{x}{x}\,dx = x\ln x - x$

$\frac{d}{dx}(x\ln x - x) = \ln x + \frac{x}{x} - 1 = \ln x$

② $\displaystyle\lim_{t\to 0} t\ln t = \lim_{t\to 0}\frac{\frac{d}{dt}\ln t}{\frac{d}{dt}\frac{1}{t}}$ (L'Hôpital)

$$= \lim_{t\to 0}\frac{1/t}{-1/t^2} = \lim_{t\to 0} -t = \underline{0}$$

§6 5: 4. $\displaystyle\int_0^1\int_0^2\int_0^{4-y^2} yz\,dz\,dy\,dx$

$$= \int_0^1\int_0^2 y(4-y^2)^2/_2 \, dy\,dx$$

$$= \int_0^1 \int_0^2 (8y - 4y^3 + y^5/_2)\,dy\,dx$$

$$= \int_0^1 (4\cdot 4 - 16 + 64/_{12})\,dx$$

$$= \frac{64}{12} = \frac{16}{3}$$

14. $\displaystyle\int_{-1}^{1}\int_0^{1-x^2}\int_0^{\sqrt{y}} x^2y^2z^2\,dz\,dy\,dx$

$V = \{(x,y,z) \mid -1 \le x \le 1,\ 0 \le y \le 1-x^2,\ 0 \le z \le \sqrt{y}\}$

a weird rounded wedge thingy. 

$$= \int_{-1}^{1}\int_0^{1-x^2} x^2y^2 y^{3/2}/_3 \, dy\,dx$$

$$= 1/_3 \int_{-1}^{1} x^2 \left(2/_9 (1-x^2)^{9/2}\right) dx$$

$$= 2/_{3^3} \int_{\pi}^{0} \cos^2\theta \sin\theta\, d\theta \qquad \left(\begin{array}{l} \cos\theta = x \\ \frac{dx}{d\theta} = -\sin\theta = \sqrt{1-x^2} \end{array}\right)$$

$(1+\cos 2\theta)(1-\cos 2\theta)^4$

$$= 2/_{3^3} \, 2\int_0^1 x^2(1-x^2)^{9/2}\,dx$$

$$= 2/_{3^3} \int_0^1 \sqrt{1-u}\, u^{9/2}\,du \qquad \left(\begin{array}{l} u = 1-x^2 \\ \frac{du}{dx} = -2x \\ x = \sqrt{1-u} \end{array}\right)$$

$$= 2/_{3^3} \int_0^1 \sqrt{1-v^2}\, v^{10}\,dv \qquad \left(\begin{array}{l} v^2 = u \\ \frac{du}{dv} = u^{1/2} \end{array}\right)$$

$$= 2/_{3^3} \int_0^{\pi/2} \sin^{10}\theta\, d\theta \qquad \left(\begin{array}{l} \sin\theta = v \\ \frac{dv}{d\theta} = \cos\theta = \sqrt{1-v^2} \end{array}\right)$$

$= 2/_{3^3}$ ... well that was needlessly indirect...

$= 2/_{3^3}$ Doable, but horrible. let's leave it like that

(Final answer is $\frac{7\pi}{(9)(2^9)}$)