2A03 Midterm 2

1. Lagrange,

$g(x,y) = x^2+y^2-4$

$\nabla f = \lambda \nabla g$

$(y+2, x) = \lambda(2x, 2y)$

$y+2 = 2\lambda x$

$x = 2\lambda y \Rightarrow 2\lambda = x/y$

$x^2+y^2=4$

$y+2 = 2\lambda x = \frac{x^2}{y}$

$y^2+2y = x^2 = 4-y^2$

$2y^2+2y-4=0$

$y^2+y-2=0$

$(y-1)(y+2)=0$

$y=1$ or $y=-2$

$y=1 \Rightarrow x = \pm\sqrt{4-1^2} = \pm\sqrt{3}$

$y=-2 \Rightarrow x=0$

so possible points for max;

$(-\sqrt{3}, 1)$, $(\sqrt{3}, 1)$, $(0,-2)$

values of f: $-3\sqrt{3}$ $\quad 3\sqrt{3}$ $\quad 0$

$3\sqrt{3}$ is global max value

2. $\underline{c}(t) = (t, \frac{2\sqrt{2}}{3}t^{3/2}, \frac{1}{2}t^2)$

$\underline{c}'(t) = (1, \sqrt{2}t^{1/2}, t)$

$\|\underline{c}'(t)\| = \sqrt{1+2t+t^2} = \sqrt{(1+t)^2} = 1+t$

$T_{\underline{c}}(t) = \underline{c}'(t)/1+t$

$T_{\underline{c}}'(t) = \frac{\underline{c}''(t)(1+t) - \underline{c}'(t)}{(1+t)^2}$

$= \frac{(-1, \frac{\sqrt{2}}{2\sqrt{t}}(1+t) - \sqrt{2t}, (1+t)-t)}{(1+t)^2}$

$= (1+t)^{-2}(-1, \frac{1}{\sqrt{2t}} + (\frac{1}{\sqrt{2}} - \sqrt{2})\sqrt{t}, 1)$

$\kappa_{\underline{c}}(t) = \frac{\|T_{\underline{c}}'(t)\|}{\|\underline{c}'(t)\|} = (1+t)^{-3}\sqrt{1+\frac{1}{2t}+2(\frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}-\sqrt{2})) + (\frac{1}{\sqrt{2}}-\sqrt{2})^2 t + 1}$

$= (1+t)^{-3}\sqrt{1+\frac{1}{2t} - 1 + \frac{t}{2} + 1}$

$(\frac{1}{\sqrt{2}} - \sqrt{2} = \frac{-1}{\sqrt{2}})$

$= \sqrt{1+\frac{1}{2t}+\frac{t}{2}+1} \Big/ (1+t)^3$

so curvature at $(1, \frac{2\sqrt{2}}{3}, \frac{1}{2})$ is

$\kappa_{\underline{c}}(1) = \frac{\sqrt{1/2+1/2+1}}{2^3} = \frac{\sqrt{2}}{8} = \frac{1}{\sqrt{32}} = 2^{-5/2}$

3. (a) Want $4(a\cos t)^2 + (b\sin t)^2 = 4$

$4a^2\cos^2 t + b^2\sin^2 t = 4$

so want $4a^2 = 4$

$b^2 = 4$

so $a=1$ $b=2$ works

(b) dist $= f(x,y) = \sqrt{x^2+y^2}$

av dist $= \int_{\underline{c}} f\,ds \Big/ \text{length}$

length $= \int_{\underline{c}} ds = \int_0^{2\pi} \|\underline{c}'(t)\|\,dt = \cancel{\int_0^{2\pi}\sqrt{\cos^2 t + 4\sin^2 t}\,dt}$

$= \int_0^{2\pi}\sqrt{\sin^2 t + 4\cos^2 t}\,dt$

$\int_{\underline{c}} f\,ds = \int_0^{2\pi}\sqrt{(\cos^2 t + 4\sin^2 t)(\sin^2 t + 4\cos^2 t)}\,dt$

$= \int_0^{2\pi}\sqrt{9\cos^2 t\sin^2 t + 4(\cos^4 t + \sin^4 t)}\,dt$

so av dist $= \frac{\int_0^{2\pi}\sqrt{9\cos^2 t\sin^2 t + 4(\cos^4 t + \sin^4 t)}\,dt}{\int_0^{2\pi}\sqrt{\sin^2 t + 4\cos^2 t}\,dt}$

4. curl $\underline{F} = \underline{0}$ ("irrotational")

$\mathbb{R}^3 =$ dom $\underline{F}$ is simply connected

so $\underline{F}$ is a gradient vector field and has path-independent integrals,

$\underline{c}(0) = (0,0,0)$ $\quad \underline{c}(1) = (1,1,1)$

Let $\gamma(t) := (t,t,t)$ $\quad t \in [0,1]$

so γ has same endpoints as $\underline{c}$

so $\int_{\underline{c}} \underline{F}\cdot d\underline{s} = \int_{\gamma} \underline{F}\cdot d\underline{s} = \int_0^1 \underline{F}(t,t,t)\cdot\gamma'(t)\,dt$

$= \int_0^1 (t^2,t^2,t^2)\cdot(1,1,1)\,dt$

$= \int_0^1 3t^2\,dt = 1$