Tutorial #1 (Sept. 15, 16)

Office hours: Wed., Fri. at 12:30 in the Math Café.

Definition. Let $f : \mathbb{R}^m \to \mathbb{R}^n$. Then

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L \quad \text{iff}$$

for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$0 < \|\vec{x} - \vec{a}\| < \delta \implies |f(\vec{x}) - L| < \varepsilon.$$

Example: (2.3, #4). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \frac{2xy}{x^2 + y^2}.$$

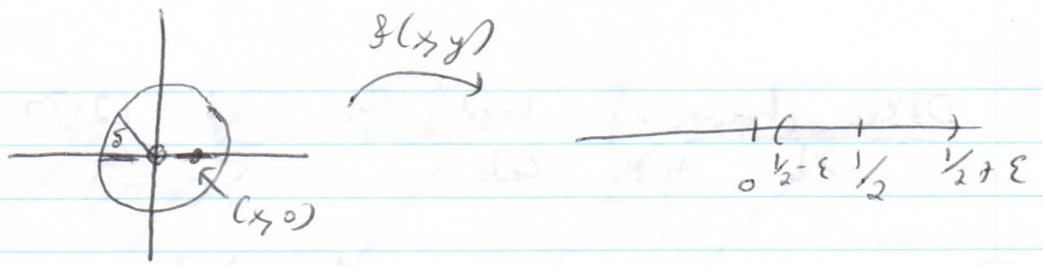
Show that $\frac{1}{2}$ cannot be the limit of f as $(x, y) \to (0, 0)$.

Let $\frac{1}{2} > \varepsilon > 0$, and consider the interval $\left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)$. We want to show

that, for every $\delta > 0$,

$$0 < \|\vec{x}\| < \delta \quad \text{but} \quad |f(\vec{x}) - \tfrac{1}{2}| \geq \varepsilon$$

for some $\vec{x}$ in the domain of f.



Consider any open ball of radius $\delta > 0$ centered at the origin. No matter how small we choose $\delta > 0$, the open ball will always contain points on the x-axis (except at $(0,0)$).

~~Any~~ At any point on the line $y=0$, $f(x,y) = 0$. So choose a point $(x,0)$ such that $(x,0)$ is in the ball of radius $\delta > 0$ and $x \neq 0$. Then we have

$$0 < \|(x,0)\| < \delta, \quad \text{but}$$

$$\cancel{\left| f(x,0) - \frac{1}{2} \right| = \left| 0 - \frac{1}{2} \right| = \frac{1}{2}}$$

$$\left| f(x,0) - \frac{1}{2} \right| = \left| 0 - \frac{1}{2} \right| = \frac{1}{2} > \varepsilon.$$

i.e. the image of $(x,0)$ under f (for any $(x,0)$ in the ball of radius $\delta > 0$ & $x \neq 0$) is never contained in the interval $\left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)$ for $0 < \varepsilon < \frac{1}{2}$. Thus the limit of f as $(x,y) \to (0,0)$ cannot be $L = \frac{1}{2}$.

(In fact, this limit doesn't exist.)

~~Recall~~

Definition. The *gradient* of a real-valued function $f : \mathbb{R}^m \to \mathbb{R}$ at $\vec{x}$ is defined to be the $1 \times m$ matrix

$$\operatorname{grad} f(\vec{x}) = \left[\frac{\partial f}{\partial x_1}(\vec{x}) \ \cdots \ \frac{\partial f}{\partial x_m}(\vec{x}) \right].$$

(We sometimes denote the gradient of f at $\vec{x}$ by $\nabla f(\vec{x})$.)

Example: (Ex. 2.40)

Define the gravitational potential function $V : \mathbb{R}^3 \to \mathbb{R}$ by

$$V(x, y, z) = \frac{-GMm}{\sqrt{x^2+y^2+z^2}}.$$

Compute $\nabla V(x, y, z)$.

(Here $G = 6.67 \times 10^{-11}\ Nm^2 kg^{-2}$ is the gravitational constant and M, m are masses of a large and small object, respectively.)

We need $\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}$.

We compute:

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\left(\frac{-GMm}{(x^2+y^2+z^2)^{1/2}}\right)$$
$$= -GMm \cdot -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x$$
$$= x \cdot \frac{GMm}{(x^2+y^2+z^2)^{3/2}}.$$

By symmetry we also get

$$\frac{\partial V}{\partial y} = y \cdot \frac{GMm}{(x^2+y^2+z^2)^{3/2}}, \quad \frac{\partial V}{\partial z} = z \cdot \frac{GMm}{(x^2+y^2+z^2)^{3/2}}.$$

So

$$\nabla V(x,y,z) = \begin{bmatrix} \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{bmatrix}$$
$$= \begin{bmatrix} x \cdot \frac{GMm}{(x^2+y^2+z^2)^{3/2}} & y \cdot \frac{GMm}{(x^2+y^2+z^2)^{3/2}} & z \cdot \frac{GMm}{(x^2+y^2+z^2)^{3/2}} \end{bmatrix}$$
$$= \frac{GMm}{(x^2+y^2+z^2)^{3/2}} \begin{bmatrix} x & y & z \end{bmatrix}.$$

(Writing $\vec{r} = (x, y, z)$, we get

$$\nabla V = \frac{GMm}{\|\vec{r}\|^3} \cdot \vec{r} \implies \vec{F} = -\nabla V,$$

where $\vec{F}$ is the gravitational force field.)

Let $F(x,y) = \dfrac{(x,y)}{\|(x,y)\|}$

$$= \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right).$$

F is a vector-valued function from $\mathbb{R}^2$ to $\mathbb{R}^2$. What is the "derivative" of F?

__Definition.__ Let $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a vector-valued function. We can write

$F(x,y) = (F_1(x,y), F_2(x,y))$.

We define the __derivative of F at (x,y)__ to be the 2×2 matrix of partial derivatives at (x,y)

$$DF(x,y) = \begin{bmatrix} \frac{\partial F_1}{\partial x}(x,y) & \frac{\partial F_1}{\partial y}(x,y) \\ \frac{\partial F_2}{\partial x}(x,y) & \frac{\partial F_2}{\partial y}(x,y) \end{bmatrix}$$

(provided that the partial derivatives here all exist).

- This will be defined in more detail and in greater generality in class.
- This matrix is sometimes called the __Jacobian matrix__ of F at (x,y).

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$F(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right) = \frac{(x,y)}{\|(x,y)\|}.$$

Calculate $DF(x,y) = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix}$.

$$\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}\left(x \cdot (x^2+y^2)^{-1/2}\right)$$

$$= (x^2+y^2)^{-1/2} + x \cdot -\frac{1}{2} \cdot 2x \cdot (x^2+y^2)^{-3/2}$$

$$= (x^2+y^2)^{-1/2} - x^2 (x^2+y^2)^{-3/2}$$

$$= \frac{(x^2+y^2) - x^2}{(x^2+y^2)^{3/2}}$$

$$= \frac{y^2}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{(x^2+y^2)^{1/2}}\right) = x \cdot -\frac{1}{2} \cdot 2y \cdot (x^2+y^2)^{-3/2}$$

$$= \frac{-xy}{(x^2+y^2)}.$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{(x^2+y^2)^{1/2}}\right) = y \cdot -\frac{1}{2} \cdot 2x \cdot (x^2+y^2)^{-3/2}$$

$$= \frac{-xy}{(x^2+y^2)^{3/2}}.$$

$$\frac{\partial F_2}{\partial y} = \frac{\partial}{\partial y}\left(y \cdot (x^2+y^2)^{-1/2}\right) = \frac{1}{(x^2+y^2)^{1/2}} + y \cdot -\frac{1}{2} \cdot 2y \cdot (x^2+y^2)^{-3/2}$$

$$= \frac{1}{(x^2+y^2)^{1/2}} - \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{x^2+y^2-y^2}{(x^2+y^2)^{3/2}} = \frac{x^2}{(x^2+y^2)^{3/2}}.$$

$$\therefore DF = \begin{bmatrix} \frac{y^2}{(x^2+y^2)^{3/2}} & \frac{-xy}{(x^2+y^2)^{3/2}} \\ \frac{-xy}{(x^2+y^2)^{3/2}} & \frac{x^2}{(x^2+y^2)^{3/2}} \end{bmatrix}$$

$$= \frac{1}{(x^2+y^2)^{3/2}} \begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix}$$

$$= \frac{1}{\|(x,y)\|} \begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix}.$$

$F(x,y) = \frac{(x,y)}{\|(x,y)\|}$ is the

vector field consists of all unit vectors pointing away from the origin.

Note that F is defined for all pairs $(x,y) \in \mathbb{R}^2$ except for $(0,0)$.

What happens at the origin?

Let $F(x,y) = \frac{1}{\|(x,y)\|}(x,y) = \left(\frac{x}{\|(x,y)\|}, \frac{y}{\|(x,y)\|}\right)$

$= \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$

Then

$$\lim_{(x,y)\to(0,0)} F(x,y) = \lim_{(x,y)\to(0,0)} \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$$

$$= \left(\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}, \lim_{(x,y)\to(0,0)} \frac{y}{\sqrt{x^2+y^2}}\right).$$

Consider $\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$.

~~[If we take the limit along $x=0$, we get~~

$$\cancel{\lim_{(x,y)\to(0,0)} \frac{0}{\sqrt{0^2+y^2}} = 0.}$$ ~~]~~

If we go along the line $y=x$, we get

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+x^2}} = \lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{2x^2}} = \lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{2}x}$$

$$= \frac{1}{\sqrt{2}}.$$

$y=0$:

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+0^2}} = \lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2}} = \lim_{(x,y)\to(0,0)} 1 = 1.$$

$\therefore \lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$ does not exist and so

$\lim_{(x,y)\to(0,0)} F(x,y)$ does not exist.