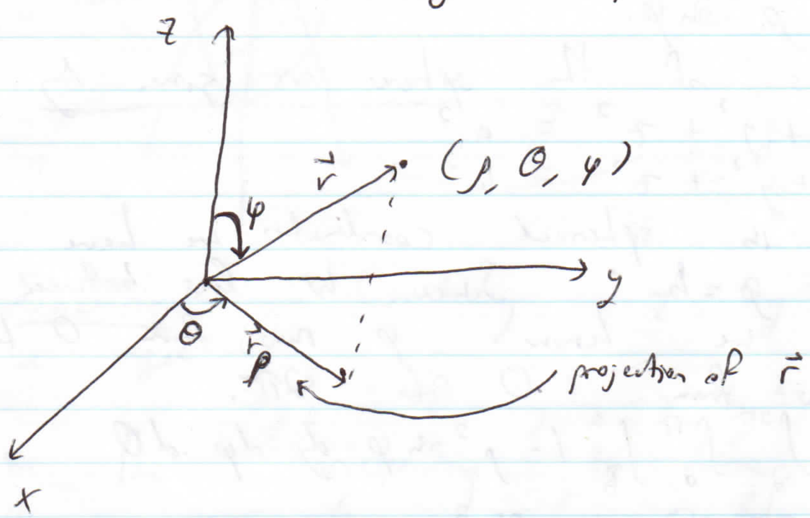
Math 2A03 Tutorial #10 (Nov. 17–18).

Definition. A point (x, y, z) in $\mathbb{R}^3$ can be represented using <u>spherical coordinates</u> by specifying the following data:

(a) The distance $\rho = \|\vec{r}\| = \sqrt{x^2+y^2+z^2}$ from the origin.

(b) The angle θ $(0 \le \theta \le 2\pi)$ in the xy-plane measured counterclockwise, between the x-axis and the projection $\vec{r}_{p}$ of $\vec{r}$ onto the xy-plane.

(c) The angle φ $(0 \le \varphi \le \pi)$ in the plane containing the z-axis and the vector $\vec{r}$, measured from the positive part of the z-axis to the negative part.



We get:

$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi.$$

<u>Example</u>: Compute the volume of the solid W lying inside the sphere of radius a and outside the sphere of radius b, where $a > b$.

To compute the volume, we use a change of variables using spherical coordinates: The Jacobian of the change of variables is given by

$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix}$$

$$= \rho^2\sin\varphi.$$

The equations of the sphere are given by

$$x^2+y^2+z^2 = a^2$$
$$x^2+y^2+z^2 = b^2$$

and so in spherical coordinates we have $\rho = a$ and $\rho = b$. Since W lies between two spheres, we know φ runs from 0 to π and θ runs from 0 to 2π.

$$\therefore \quad \mathrm{vol}(W) = \int_0^{2\pi}\int_0^{\pi}\int_b^a 1\cdot\rho^2\sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\sin\varphi\int_b^a\rho^2\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\sin\varphi\left(\frac{\rho^3}{3}\right)\Big|_b^a d\varphi\, d\theta$$

$$= \frac{1}{3}(a^3-b^3)\int_0^{2\pi} -\cos\varphi\Big|_0^{\pi} d\theta$$

$$= \frac{1}{3}(a^3-b^3)\int_0^{2\pi}\left(-\cos\pi - (-\cos 0)\right) d\theta$$

$$= \frac{1}{3}(a^3-b^3)\int_0^{2\pi}(1+1)\, d\theta$$

$$= \frac{2}{3}(a^3-b^3)\int_0^{2\pi} d\theta$$

$$= \frac{2}{3}(a^3-b^3)\ \theta\Big|_0^{2\pi}$$

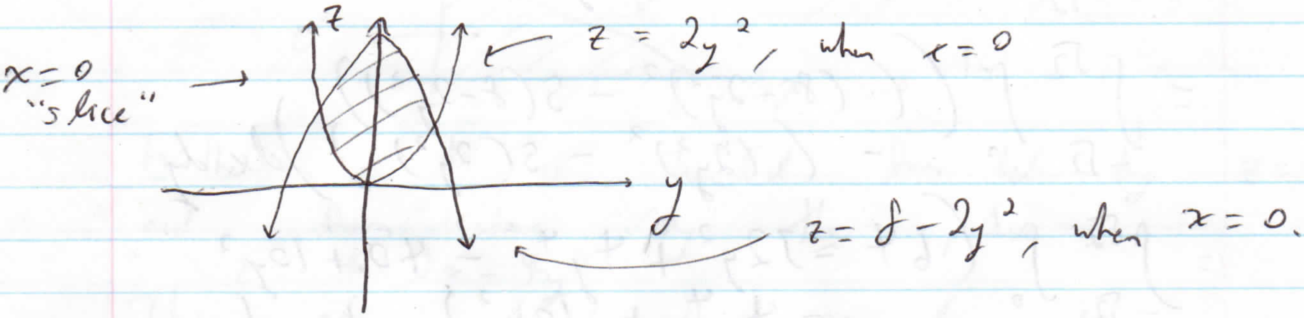
$$= \frac{4\pi}{3}(a^3-b^3).$$

Note that if $b=0$, we get the volume of the sphere of radius a:

$$\mathrm{vol}(W) = \frac{4\pi}{3} a^3,$$

which agrees with the usual formula.

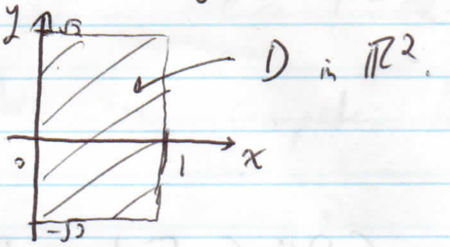
Exercise: (6.5, #5) Compute the triple integral $\iiint_W f\,dV$, where $f(x,y,z) = 2z - 5$ and W is the three-dimensional solid between the surfaces $z = 2y^2$ and $z = 8 - 2y^2$ for $0 \le x \le 1$.



Solution: Combining $z = 2y^2$ and $z = 8 - 2y^2$, we get

$$2y^2 = 8 - 2y^2 \Rightarrow 4y^2 = 8 \Rightarrow y = \pm\sqrt{2},$$

so the corresponding two-dimensional region in the xy-plane looks like the rectangle



By the first picture above, we know that the bottom surface is $z = 2y^2$ and the top surface is $z = 8 - 2y^2$. We can compute $\iiint_W f\,dV$ as the following iterated integral:

$$\iiint_W (2z-5)\,dV$$

$$= \iint_D \left(\int_{2y^2}^{8-2y^2} 2z-5 \; dz \right) dA$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \int_0^1 \int_{2y^2}^{8-2y^2} 2z-5 \; dz \; dx \; dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \int_0^1 (z^2-5z)\Big|_{2y^2}^{8-2y^2} dx \; dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \int_0^1 \Big(\left((8-2y^2)^2 - 5(8-2y^2) \right) - \left((2y^2)^2 - 5(2y^2) \right) \Big) dx \; dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \int_0^1 \big(64 - 32y^2 + 4y^4 - 40 + 10y^2 - 4y^4 + 10y^2 \big) \; dx \; dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \int_0^1 (24 - 12y^2) \; dx \; dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} (24-12y^2)\, x\Big|_0^1 \; dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} (24-12y^2) \; dy$$

$$= 24y - 4y^3 \Big|_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \left(24\sqrt{2} - 4(2)^{3/2}\right) - \left(24(-\sqrt{2}) - 4(-2^{3/2})\right)$$

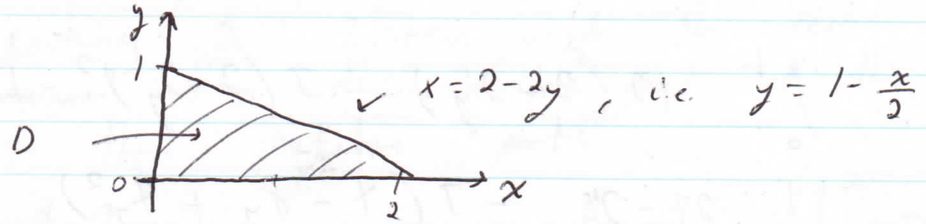
$$= 24\sqrt{2} + 24\sqrt{2} - 4(2)^{3/2} - 4(2)^{3/2}$$

$$= 48\sqrt{2} - 8(\sqrt{2})^3$$

Exercise: (6.5, #11) Evaluate the following iterated integral and describe the region of integration:

$$\int_0^1 \int_0^{2-2y} \int_0^{4-2x-4y} 3 \, dz \, dx \, dy.$$

Solution: Let W be the solid which we are integrating over. Since $0 \leq y \leq 1$ and $0 \leq x \leq 2-2y$, the corresponding two-dimensional region D in the xy-plane is given by



Furthermore W is bounded from below by $z=0$ and from above by the plane

$$z = 4 - 2x - 4y.$$

This plane intersects the xy-plane when

$$0 = 4 - 2x - 4y \Rightarrow x = 2 - 2y,$$

which is just the line shown above. Thus the solid W is just the region in the first octant in $\mathbb{R}^3$ bounded from above by

$$z = 4 - 2x - 4y.$$

We can compute the iterated integral as follows:

$$\int_0^1 \left(\int_0^{2-2y} \left(\int_0^{4-2x-4y} 3\,dz \right) dx \right) dy$$

$$= \int_0^1 \left(\int_0^{2-2y} (3z)\Big|_0^{4-2x-4y} dx \right) dy$$

$$= \int_0^1 \left(\int_0^{2-2y} (12-6x-12y)\, dx \right) dy$$

$$= \int_0^1 (12x - 3x^2 - 12xy)\Big|_0^{2-2y} dy$$

$$= \int_0^1 12(2-2y) - 3(2-2y)^2 - 12(2-2y)y \; dy$$

$$= \int_0^1 24 - 24y - 3(4 - 8y + 4y^2) - 24y + 24y^2 \; dy$$

$$= \int_0^1 24 - 24y - 12 + 24y - 12y^2 - 24y + 24y^2 \, dy$$

$$= \int_0^1 12 + 12y^2 - 24y \; dy$$

$$= 12y + 4y^3 - 12y^2 \Big|_0^1$$

$$= 12 + 4 - 12 = 4.$$