Math 2A03 Tutorial #11 (Nov. 24-25) C

Exercise: Find a parametrization of the following surfaces in $\mathbb{R}^3$:

(i) The part of the plane $z - 3y + x = 2$ inside the cylinder $x^2 + y^2 = 4$.

(ii) The part of the plane $x + 2y + z = 6$ in the first octant.

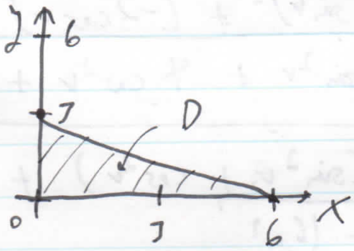
Solution: (i) Parametrize the plane as the graph of the surface $z = 2 - x + 3y$ to get a parametrization $\vec{r}(u,v) = (u, v, 2 - u + 3v)$. Since we only need points on the above plane which lie inside the cylinder $x^2 + y^2 = 4$, our domain D consists of the points (u,v) such that $u^2 + v^2 = 4$. $\therefore$ The required parametrization $\vec{r} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is given by

$$\vec{r}(u,v) = (u, v, 2 - u + 3v), \quad (u,v) \in D$$

where D is the disk $u^2 + v^2 \leq 4$ of radius 2.

(ii) Parametrize the plane as the graph of the function $z = 6 - x - 2y$ to get a parametrization $\vec{r}(u,v) = (u, v, 6 - u - 2v)$. Since the domain of parametrization consists of points in the first octant, we need $x \geq 0$, $y \geq 0$ and $6 - x - 2y \geq 0$, i.e. $x + 2y \leq 6$.

Hence the region $D \subseteq \mathbb{R}^2$ of parametrization is the triangular region bounded by the lines $x = 0$, $y = 0$ and $x + 2y = 6$ (i.e. $y = 3 - \frac{x}{2}$):



Hence the required parametrization is given by

$$\vec{r}(u,v) = (u, v, 6 - u - 2v), \quad (u,v) \in D$$

where D is the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(6,0)$ and $(0,3)$.

Exercise: (7.3 #17). Compute the surface area of the part of the surface
$$\vec{r}(u,v) = (2u\cos v, 2u\sin v, v),$$
where $0 \le u \le 2$, $0 \le v \le \pi$.

Solution: From the above parametrization we can compute the tangent vectors:
$$\vec{T}_u = (2\cos v, 2\sin v, 0)$$
$$\vec{T}_v = (-2u\sin v, 2u\cos v, 1)$$
and so the surface normal $\vec{N}$ is given by:
$$\vec{N} = \vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\cos v & 2\sin v & 0 \\ -2u\sin v & 2u\cos v & 1 \end{vmatrix}$$
$$= \hat{i}\begin{vmatrix} 2\sin v & 0 \\ 2u\cos v & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 2\cos v & 0 \\ -2u\sin v & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 2\cos v & 2\sin v \\ -2u\sin v & 2u\cos v \end{vmatrix}$$
$$= \hat{i}(2\sin v) - \hat{j}(2\cos v) + \hat{k}(4u\cos^2 v + 4u\sin^2 v)$$
$$= (2\sin v, -2\cos v, 4u(\cos^2 v + \sin^2 v))$$
$$= (2\sin v, -2\cos v, 4u).$$

Hence
$$\|\vec{N}\| = \|(2\sin v, -2\cos v, 4u)\|$$
$$= \sqrt{(2\sin v)^2 + (-2\cos v)^2 + (4u)^2}$$
$$= \sqrt{4\sin^2 v + 4\cos^2 v + 16u^2}$$
$$= \sqrt{4(\sin^2 v + \cos^2 v) + 16u^2}$$
$$= \sqrt{4 + 16u^2}$$
$$= \sqrt{4(1 + 4u^2)}$$
$$= 2 \cdot \sqrt{1 + 4u^2}.$$

$\therefore$ the surface area is

$$\iint_S dS = \iint_D 2\sqrt{1+4u^2}\, dA.$$

D is the region in the uv-plane given by $0 \le u \le 2$ and $0 \le v \le \pi$.

$$\therefore \iint_S dS = \int_0^2 \left(\int_0^\pi 2\sqrt{1+4u^2}\, dv \right) du$$

$$= 2\int_0^2 \sqrt{1+4u^2} \left(\int_0^\pi dv \right) du$$

$$= 2 \int_0^2 \sqrt{1+4u^2} \cdot \pi \, du$$

$$= 2\pi \int_0^2 \sqrt{1+4u^2}\, du$$

$$\simeq 29.1966$$

(to compute $\int_0^2 \sqrt{1+4u^2}$, we either an integration table or trigonometric substitution ...)

Exercise: (7.4 #13) Compute $\iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F}(x,y,z) = (x,y,z)$ and S is the surface parametrized by $\vec{r}(u,v) = (e^u\cos v, e^u\sin v, v)$, $0 \le u \le \ln 2$, $0 \le v \le \pi$, oriented with an upward-pointing normal.

Solution: Compute the tangent vectors to be

$\vec{T}_u = (e^u\cos v, e^u\sin v, 0)$, $\vec{T}_v = (-e^u\sin v, e^u\cos v, 1)$

and so the surface normal is

$$\vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ e^u\cos v & e^u\sin v & 0 \\ -e^u\sin v & e^u\cos v & 1 \end{vmatrix} = \hat{i}\begin{vmatrix} e^u\sin v & 0 \\ e^u\cos v & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} e^u\cos v & 0 \\ -e^u\sin v & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} e^u\cos v & e^u\sin v \\ -e^u\sin v & e^u\cos v \end{vmatrix}$$

$$= (e^u\sin v, -e^u\cos v, e^{2u}\cos^2 v + e^{2u}\sin^2 v)$$
$$= (e^u\sin v, -e^u\cos v, e^{2u}).$$

This normal is upward-pointing as needed, so we can proceed.

$\vec{F}(\vec{r}(u,v)) = (e^u\cos v, e^u\sin v, v)$, so

$$\vec{F}(\vec{r}(u,v))\cdot\vec{N} = e^{2u}\cos v\sin v - e^{2u}\cos v\sin v + e^{2u}v = e^{2u}v.$$

Since $0 \le u \le \ln 2$ and $0 \le v \le \pi$, we let D be the corresponding rectangular region in the uv-plane.

$$\therefore \iint_S (x,y,z)\cdot d\vec{S} = \iint_D e^{2u}v\,dA$$
$$= \int_0^{\ln 2}\left(\int_0^{\pi} e^{2u}v\,dv\right)du$$
$$= \int_0^{\ln 2} e^{2u}\left(\frac{v^2}{2}\Big|_0^{\pi}\right)du$$
$$= \frac{\pi^2}{2}\left(\frac{e^{2u}}{2}\Big|_0^{\ln 2}\right)$$
$$= \frac{\pi^2}{2}\cdot\left(\frac{e^{2\ln 2}}{2} - \frac{e^0}{2}\right)$$
$$= \frac{\pi^2}{2}\left(2 - \frac{1}{2}\right)$$
$$= \frac{\pi^2}{2}\cdot\frac{3}{2} = \frac{3\pi^2}{4}.$$

Exercise: (7.4, #19). Compute the flux of $\vec{F} = x\hat{i}$ out of the closed region S bounded by the paraboloids $z = x^2+y^2$ and $z = 12-x^2-y^2$.

Solution: Let S_1 be the paraboloid $z = x^2+y^2$ and let S_2 be $z = 12-x^2-y^2$. Combining the two equations we get

$$x^2+y^2 = 12-x^2-y^2$$
$$\Rightarrow \quad 2x^2+2y^2 = 12$$
$$\Rightarrow \quad x^2+y^2 = 6.$$

Hence the region of integration D in $\mathbb{R}^2$ (for both surfaces) is the disk $x^2+y^2 \le 6$. The flux out of S is given by the sum of $\iint_{S_1} \vec{F}\cdot d\vec{S}$ and $\iint_{S_2} \vec{F}\cdot d\vec{S}$.

polar coordinates: $0 \le r \le \sqrt{6}$, $0 \le \theta \le 2\pi$.

First we compute $\iint_{S_1} \vec{F}\cdot d\vec{S}$:

Parametrize S_1 by $\vec{r}_1(u,v) = (u, v, u^2+v^2)$. Then $\vec{T_u} = (1, 0, 2u)$, $\vec{T_v} = (0, 1, 2v)$ and so

$$\vec{N}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = (-2u, -2v, 1).$$

$\vec{N}$ points inwards into the paraboloid. We want the outward normal so we compute the surface integral as usual and then change the sign of the result:

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}_1(u,v)) \cdot \vec{N}_1 \, dA$$
$$= \iint_D (u, 0, 0)\cdot(-2u, -2v, 1)\, dA$$
$$= -2 \iint_D u^2 \, dA.$$

polar coordinates ($u = r\cos\theta$, $v = r\sin\theta$) $\rightarrow$

$$= -2\int_0^{2\pi}\left(\int_0^{\sqrt{6}} r^3 \cos^2\theta \, dr\right) d\theta$$
$$= -18\pi.$$

Since we need to change the sign of the result, we get

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = 18\pi.$$

So the outward flow is 18π.

Now parametrize S_2 by $\vec{r}_2(u, v) = (u, v, 12 - u^2 - v^2)$. Then

$$\vec{N}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = (2u, 2v, 1).$$

$\vec{N}_2$ points outwards, so we can proceed as usual.

$$\therefore \iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_D (u, 0, 0) \cdot (2u, 2v, 1)\, dA$$

$$= 2 \cdot \iint_D u^2 \, dA.$$

Using the same calculation as for $\iint_{S_1} \vec{F} \cdot d\vec{S}$, we can compute

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = 18\pi.$$

$$\therefore \iint_S \vec{F} \cdot d\vec{S} = 18\pi + 18\pi$$

$$= 36\pi.$$

So the total flux across S is 36π.