Math 2A03 Tutorial 2 (Sept. 22 - 23)

Directional derivatives:

Definition. Let $f: U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real-valued differentiable function. The directional derivative of f at a point $\vec{p} = (x_1, \dots, x_n) \in U$ in the direction of the unit vector $\vec{u} = (u_1, \dots, u_n)$ is given by

$$D_{\vec{u}} f(\vec{p}) = \frac{d}{dt} f(\vec{p} + t\vec{u}) \Big|_{t=0},$$

$$= \lim_{t \to 0} \frac{f(\vec{p} + t\vec{u}) - f(\vec{p})}{t}.$$

Note that the directional derivative is a generalization of the partial derivative; the directional derivative gives the rate of change in any specified direction. Indeed, if we let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$, $\vec{u} = \hat{\imath} = (1, 0)$, and $\vec{p} = (a, b) \in \mathbb{R}^2$, we see that

$$D_{\hat{\imath}} f(a, b) = \lim_{t \to 0} \frac{f((a,b) + t(1,0)) - f(a,b)}{t}$$

$$= \lim_{t \to 0} \frac{f(a+t, b) - f(a, b)}{t}$$

$$= \frac{\partial f}{\partial x}(a, b).$$

Similarly, if we let $\vec{u} = \hat{\jmath} = (0, 1)$, then

$$D_{\hat{\jmath}} f(a,b) = \frac{\partial f}{\partial y}(a, b).$$

<u>Theorem.</u> Let $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, and let $\vec{p} = (x_1, \dots, x_n) \in U$. Then for any unit vector $\vec{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$ we have

$$D_{\vec{u}} f(\vec{p}) = \nabla f(\vec{p}) \cdot \vec{u}.$$

<u>Exercise</u>: (2.7, #25) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = 2xy$. In what direction(s) at the point $(1,2)$ is the direction derivative of f equal to 4?

<u>Solution</u>: Let $\vec{u} = (a,b)$ be a unit vector in $\mathbb{R}^2$. Since $\vec{u}$ is a unit vector

$\|\vec{u}\| = \sqrt{a^2+b^2} = 1$ must ~~hold~~ hold, i.e.

$$a^2 + b^2 = 1.$$

By the theorem, we know that

$$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u}$$
$$= \nabla f(1,2) \cdot (a,b).$$

Since $\nabla f(1,2) = (2y, 2x)\big|_{(x,y)=(1,2)} = (4,2)$, we see that

$$4a + 2b = 4$$

must hold in order for $D_{\vec{u}} f(1,2) = 4$ to hold.

i.e. $b = 2 - 2a$. Substituting this in for b in the equation $a^2+b^2=1$ yields

$$1 = a^2 + (2-2a)^2 = a^2 + 4 + 4a^2 - 8a$$

i.e. $5a^2 - 8a + 3 = 0$. Solving for a, we get $a = \frac{3}{5}$ or $a = 1$, respectively this implies $b = 4/5$ or $b = 0$. $\therefore D_{\vec{u}} f(1,2) = 4$ if $\vec{u} = (3/5, 4/5)$ or if $\vec{u} = (1,0) = \hat{i}$.

The Chain Rule:

Theorem (Chain Rule). Suppose $\vec{F} : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is a function differentiable at $\vec{a} \in U$ (where U is a open set in $\mathbb{R}^m$), $\vec{G} : V \subseteq \mathbb{R}^n \to \mathbb{R}^p$ is a function differentiable at $\vec{F}(\vec{a}) \in V$ (where V is an open set in $\mathbb{R}^n$), and $\vec{F}(U) \subseteq V$. Then the composition $\vec{G} \circ \vec{F} : \mathbb{R}^m \to \mathbb{R}^p$ is differentiable at $\vec{a}$ and

$$D(\vec{G} \circ \vec{F})(\vec{a}) = D\vec{G}(\vec{F}(\vec{a})) \cdot D\vec{F}(\vec{a}).$$

Special case: If $\vec{c}$ is a path in $\mathbb{R}^3$ (i.e. $\vec{c} : \mathbb{R} \to \mathbb{R}^3$ is given by $\vec{c}(t) = (x(t), y(t), z(t))$) and $f : \mathbb{R}^3 \to \mathbb{R}$ is a real-valued function (such that both $\vec{c}$ and f are differentiable), then

$$(f \circ \vec{c})(t) = f(\vec{c}(t)) = f(x(t), y(t), z(t))$$

and so applying the chain rule we get

$$D(f \circ \vec{c})(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \nabla f(\vec{c}(t)) \cdot \vec{c}\,'(t). \qquad (*)$$

Exercise: (#7, 2.6) Let f and $\vec{c}$ be defined by

$$f(x,y,z) = xy + \cos(x^2+z^2),$$
$$\vec{c}(t) = (t\sin t,\ t,\ t\cos t).$$

Compute $(f\circ\vec{c})'(t)$ first by computing the derivative of the composition, ~~the~~ and then by using $(*)$.

Solution: First we compute the derivative by using the composition $f\circ\vec{c}$:

$$(f\circ\vec{c})(t) = f(\vec{c}(t)) = f(t\sin t,\ t,\ t\cos t)$$
$$= t^2\sin t + \cos(t^2),$$

and so

$$(f\circ\vec{c})'(t) = 2t\sin t + t^2\cos t - 2t\sin(t^2).$$

Using the formula given by $(*)$ we get:

$$\begin{aligned}
D(f\circ\vec{c})(t) &= \nabla f(\vec{c}(t))\cdot\vec{c}'(t)\\
&= \left(y - 2x\sin(x^2+z^2),\ x,\ -2z\sin(x^2+z^2)\right)\Big|_{(t\sin t,\ t,\ t\cos t)}\\
&\quad\cdot\left(\sin t + t\cos t,\ 1,\ \cos t - t\sin t\right)\\
&= \left(t - 2(t\sin t)\sin(t^2),\ t\sin t,\ -2(t\cos t)\sin(t^2)\right)\\
&\quad\cdot\left(\sin t + t\cos t,\ 1,\ \cos t - t\sin t\right)\\
&= t\sin t + t^2\cos t - 2t\sin^2 t\sin(t^2) - 2t^2\sin t\sin(t^2)\cos t\\
&\quad + t\sin t - 2t\cos^2 t\sin(t^2) + 2t^2\cos t\sin(t^2)\sin t\\
&= 2t\sin t + t^2\cos t - 2t\sin(t^2)(\sin^2 t + \cos^2 t)\\
&= 2t\sin t + t^2\cos t - 2t\sin(t^2).
\end{aligned}$$

[...]rcise: (# 21, 2.6) Let $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^3$ and $g: \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$\vec{F}(x,y) = (e^x, xy, e^y),$$

$$g(u,v,w) = uw + v^2.$$

Compute $D(g \circ \vec{F})(0,0)$.

Solution: By the chain rules we know

$$D(g\circ\vec{F})(0,0) = Dg(\vec{F}(0,0)) \cdot D\vec{F}(0,0).$$

$$Dg(\vec{F}(0,0)) = \begin{bmatrix} w & 2v & u \end{bmatrix}_{\vec{F}(0,0)}$$

$$= \begin{bmatrix} w & 2v & u \end{bmatrix}_{(1,0,1)}$$

$$= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}.$$

$$D\vec{F}(0,0) = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} \end{bmatrix}_{(0,0)} = \begin{bmatrix} e^x & 0 \\ y & x \\ 0 & e^y \end{bmatrix}_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\therefore \quad D(g\circ\vec{F})(0,0) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Exercise: Use the chain rule to prove the dot product rule in two dimensions:

If $\vec{v}, \vec{w} : U \subseteq \mathbb{R} \to \mathbb{R}^2$ are differentiable at $a \in U$, then their dot product $\vec{v} \cdot \vec{w}$ is differentiable at a and

$$(\vec{v} \cdot \vec{w})'(a) = \vec{v}'(a) \cdot \vec{w}(a) + \vec{v}(a) \cdot \vec{w}'(a).$$

Proof: Write $\vec{v}(t) = (v_1(t), v_2(t))$ and $\vec{w}(t) = (w_1(t), w_2(t))$ for real-valued functions v_1, v_2, w_1, w_2 on U, and define functions $f : \mathbb{R} \to \mathbb{R}^4$, $g : \mathbb{R}^4 \to \mathbb{R}$ by

$$f(t) = (v_1(t), v_2(t), w_1(t), w_2(t)),$$

$$g(t_1, t_2, t_3, t_4) = t_1 t_3 + t_2 t_4$$
$$= (t_1, t_2) \cdot (t_3, t_4).$$

By the chain rule,

$$D(g \circ f)(t) = Dg(f(t)) \cdot Df(t).$$

We compute:

$$Df(t) = \begin{bmatrix} v_1'(t) \\ v_2'(t) \\ w_1'(t) \\ w_2'(t) \end{bmatrix},$$

$$Dg(f(t)) = \begin{bmatrix} t_3 & t_4 & t_1 & t_2 \end{bmatrix}_{(v_1(t), v_2(t), w_1(t), w_2(t))}$$
$$= \begin{bmatrix} w_1(t) & w_2(t) & v_1(t) & v_2(t) \end{bmatrix}.$$

$$\therefore \underbrace{D(g \circ f)(t)}_{\vec{v} \cdot \vec{w}} = \begin{bmatrix} w_1(t) & w_2(t) & v_1(t) & v_2(t) \end{bmatrix} \cdot \begin{bmatrix} v_1'(t) \\ v_2'(t) \\ w_1'(t) \\ w_2'(t) \end{bmatrix}$$

$$= v_1'(t) w_1(t) + v_2'(t) w_2(t)$$
$$+ v_1(t) w_1'(t) + v_2(t) w_2'(t)$$
$$= (v_1(t), v_2(t))' \cdot (w_1(t), w_2(t))$$
$$+ (v_1(t), v_2(t)) \cdot (w_1(t), w_2(t))'$$
$$= \vec{v}'(t) \cdot \vec{w}(t) + \vec{v}(t) \cdot \vec{w}'(t).$$