Math 2A03 Tutorial #4 (Oct. 6-7)

Theorem 4.12. Let $f, g : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be C^1 functions. If $f(x,y)$ has a local extrema subject to the constraint $g(x,y) = k \in \mathbb{R}$ at the point (x_0, y_0), and if $\nabla g(x_0, y_0) \neq 0$, then $\nabla f(x_0, y_0) = \lambda \cdot \nabla g(x_0, y_0)$ for some $\lambda \in \mathbb{R}$.

Example: Let $T(x,y) = x^2 - y + 200$ be a function for the temperature at a point (x,y) on a thin metal plate in the shape of the unit disk $D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$. Find the maximum temperature on the rim of D.

i.e. we have to find the maximum of $T(x,y)$ among the points which satisfy $x^2+y^2=1$. So let $g(x,y) = x^2+y^2$; our constraint is then given by $g(x,y)=1$. Using the above theorem, we can find the maximum of $T(x,y)$ subject to the constraint $g(x,y)=1$.

We know

$$\nabla T(x,y) = \lambda \nabla g(x,y)$$

holds necessarily for any local extrema of f.

$\nabla T = (2x, -1)$ and $\nabla g = (2x, 2y)$ implies

$$(2x, -1) = \lambda \cdot (2x, 2y)$$

$$\Rightarrow \quad 2x = 2x \cdot \lambda$$
$$-1 = 2y \cdot \lambda$$

$$\Rightarrow \quad 2x \cdot (1-\lambda) = 0,$$

so either $x=0$ or $\lambda = 1$. If $x=0$, then $g(x,y) = x^2+y^2 = 1$ implies $y = \pm 1$. So we get two candidates for extreme values: The point $(0,1)$ or $(0,-1)$.

If, on the other hand, $\lambda = 1$, then by the equation $-1 = 2y \cdot \lambda$ we get $y = -\frac{1}{2}$. Applying the constraint $x^2 + y^2 = 1$ again, we see that

$$x^2 + \left(-\frac{1}{2}\right)^2 = 1$$
$$\Rightarrow \quad x^2 = \frac{3}{4} \quad \Rightarrow \quad x = \pm \frac{\sqrt{3}}{2}.$$

So we get two more candidates: $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. Hence we have four candidates for extreme values of T subject to $g(x,y) = 1$:

$$(0,1),\ (0,-1),\ \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right),\ \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right).$$

It remains to check that $\nabla g(x,y) \neq 0$ for all (x,y) satisfying the constraint $g(x,y) = 1$. But if $\nabla g(x,y) = (2x, 2y) = 0$, $x = y = 0$ must hold and so such a point does not satisfy the constraint $g(x,y) = x^2 + y^2 = 1$. So $\nabla g(x,y) \neq 0$ for all (x,y) lying on the constraint curve $g(x,y) = 1$.

Now we evaluate T at all four points above. We get

$$T(0,1) = 199,$$
$$T(0,-1) = 201,$$
$$T\left(\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}\right) = 201.25, \quad \text{and}$$
$$T\left(-\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}\right) = 201.25.$$

Thus the maximum of $T(x,y)$ on the circle $x^2 + y^2 = 1$ is reached at the points $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, $\left(\frac{-\sqrt{3}}{2}, -\frac{1}{2}\right)$.

<u>The method of Lagrange multipliers</u> (two variables):
To find extreme values of a C^1 function $f(x,y)$ subject to a C^1 constraint $g(x,y)=k$, we identify all points (x_0,y_0) that satisfy any of the following:
(1) $\nabla f(x_0,y_0) = \lambda \cdot \nabla g(x_0,y_0)$ for $\lambda \in \mathbb{R}$ and $g(x_0,y_0)=k$;
(2) $\nabla g(x_0,y_0) = 0$ and $g(x_0,y_0)=k$;
(3) (x_0,y_0) is an endpoint of the curve $g(x,y)=k$ (if such endpoints exist).

If the constraint curve $g(x,y)=k$ is a closed bounded subset of $\mathbb{R}^2$, then the largest of the values $f(x_0,y_0)$ for points (x_0,y_0) satisfying one of the above is the maximum value of f subject to $g(x,y)=k$. Similarly, the smallest value is the minimum.

If $g(x,y)=k$ is not closed or not bounded we may need additional arguments to determine if the point (x_0,y_0) is a maximum or a minimum (or neither).

<u>Exercise</u>: (4.4, #11). Find the extreme values of $f(x,y)=3xy$ subject to the constraint $x^2+y^2=4$.
<u>Solution</u>: Let $g(x,y)=x^2+y^2$; then the constraint $g(x,y)=4$ is a circle in $\mathbb{R}^2$ and hence a closed and bounded subset of $\mathbb{R}^2$.
We have $\nabla f = \lambda \nabla g$ and so we compute:

$$\nabla f(x,y) = (3y, 3x)$$
$$\nabla g(x,y) = (2x, 2y),$$

so we get the following system of equations:

$$3y = \lambda 2x$$
$$3x = \lambda 2y$$
$$x^2+y^2=4.$$

This is equivalent to the system

$$2\lambda = 3y/x,$$
$$2\lambda = 3x/y,$$
$$x^2+y^2=4.$$

Thus $\frac{3y}{x} = \frac{3x}{y}$, i.e. $x^2=y^2$. Substituting in for $x^2+y^2=4$ we get $2x^2=4$, i.e. $x=\pm\sqrt{2}$. If $x=\sqrt{2}$ then $y=\pm\sqrt{2}$ by the constraint and if $x=-\sqrt{2}$ then $y=\pm\sqrt{2}$. So we have four candidate points for extreme values: $(\sqrt{2},\sqrt{2})$, $(-\sqrt{2},\sqrt{2})$, $(\sqrt{2},-\sqrt{2})$ and $(-\sqrt{2},-\sqrt{2})$. So we have all points corresponding to part (1) above.

Now if $\nabla g(x,y)=0$ then $2x=0$ and $2y=0$, i.e. $x=y=0$. ~~Thus~~ However, the point $(0,0)$ does not satisfy $x^2+y^2=4$ and so no point satisfies (2).

Finally note that the constraint curve is a circle and so it has no endpoints. Thus (3) does not give us any candidates for constrained extrema.

Since $x^2+y^2=4$ determines a closed and bounded set in $\mathbb{R}^2$, the constrained extrema of f are given by the maximum and minimum values of f at the four points above. So compute:

$$f(\sqrt{2},\sqrt{2}) = 6$$
$$f(-\sqrt{2},\sqrt{2}) = -6$$
$$f(\sqrt{2},-\sqrt{2}) = -6$$
$$f(-\sqrt{2},-\sqrt{2}) = \cancel{3} = 6$$

Thus the maximum value of f occurs at the points $(\sqrt{2},\sqrt{2})$, $(-\sqrt{2},-\sqrt{2})$ while the minimum of f occurs at the points $(-\sqrt{2},\sqrt{2})$, $(\sqrt{2},-\sqrt{2})$.

Example where the constraint is not a bounded set:
Find the extrema of $f(x,y) = x+y$ subject to the constraint $xy=1$.

Let $g(x,y) = xy$. Since $\nabla f = \lambda \nabla g$ must hold, we get

$$(1,1) = \lambda (y,x)$$

$$\Rightarrow \quad 1 = \lambda y$$

$$1 = \lambda x$$

$$xy = 1.$$

Since the constraint curve $y = \frac{1}{x}$ is a hyperbola it determines an unbounded set in $\mathbb{R}^2$. So we are not guaranteed the existence of absolute extrema. But they may still exist: we proceed as usual.

Going back to the above linear system, we see that $x=y$ from the first two equations since $\lambda x = \lambda y$ implies $x=y$. From the constraint $xy=1$, we get $x^2=1$ and so $x = \pm 1$. Hence $y = \pm 1$ as well. So we get ~~the~~ two potential points which are candidates for extreme values of f: $(1,1)$ and $(-1,-1)$. Note that

$$f(1,1) = 2 \qquad f(-1,-1) = -2.$$

But since our constraint curve is unbounded, these might not be the extreme values we are looking for. Indeed, consider the point $(2, \frac{1}{2})$: This point satisfies the constraint $xy=1$, and

$$f(2, \tfrac{1}{2}) = 2 + \frac{1}{2} = \frac{5}{2} > 2 = f(1,1),$$

so f is not a constrained maximum of f. Similarly

$$f(-2, -\tfrac{1}{2}) = -2 - \frac{1}{2} = -\frac{5}{2} < -2 = f(-1,-1)$$

and so $(-1,-1)$ is not a constrained minimum of f.