Definition. Let $\vec{c} : [a,b] \to \mathbb{R}^2$ be a C^1 path. The *arc-length function* $s(t)$ of $\vec{c}(t)$ is

$$s(t) = \int_a^t \|\vec{c}\,'(\tau)\|\, d\tau.$$

If we replace the parametrization $\vec{c}(t)$ by the parametrization $\vec{c}(s)$, we call $\vec{c}(s)$ the *parametrization by arc-length*.

Exercise: (3.3, #27, 29). Find the arc-length function $s(t)$ of each given path $\vec{c}(t)$ and reparametrize each path $\vec{c}(t)$ by its arc-length:

(i) $\vec{c}(t) = (5\cos t, 5\sin t, 12t)$, $t \in [0, \frac{\pi}{4}]$.

(ii) $\vec{c}(t) = (e^t \cos t, e^t \sin t)$, $t \in [0,1]$.

Solution: (i) First we compute $\vec{c}\,'(t)$:

$$\vec{c}\,'(t) = (-5\sin t, 5\cos t, 12),$$

and so

$$\begin{aligned}\|\vec{c}\,'(t)\| &= \sqrt{(-5\sin t)^2 + (5\cos t)^2 + 12^2} \\ &= \sqrt{25\sin^2 t + 25\cos^2 t + 144} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13.\end{aligned}$$

$$\begin{aligned}\therefore\quad s(t) &= \int_0^t \|\vec{c}\,'(\tau)\|\, d\tau \\ &= \int_0^t 13\, d\tau \\ &= 13\tau \Big|_0^t \\ &= 13t,\end{aligned}$$

where $t \in [0, \frac{\pi}{4}]$.

So we have $s = 13t$, i.e. $t = \frac{s}{13}$.
If we parametrize the path $\vec{c}$ by the arc-length function s, we get

$$\vec{c}(s) = \left(5\cos\left(\frac{s}{13}\right), 5\sin\left(\frac{s}{13}\right), \frac{12}{13}s\right).$$

Since originally we had $t \in [0, \pi/4]$, we now have $s \in [0, \frac{13\pi}{4}]$.

(ii)
$$\begin{aligned}\vec{c}\,'(t) &= (e^t\cos t - e^t\sin t, e^t\sin t + e^t\cos t)\\ &= (e^t(\cos t - \sin t), e^t(\sin t + \cos t))\\ &= e^t(\cos t - \sin t, \sin t + \cos t),\end{aligned}$$

so
$$\begin{aligned}\|\vec{c}\,'(t)\| &= \|e^t(\cos t - \sin t, \sin t + \cos t)\|\\ &= e^t \cdot \|(\cos t - \sin t, \sin t + \cos t)\|\\ &= e^t \cdot \sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2}\\ &= e^t \cdot \sqrt{\cos^2 t - 2\cos t\sin t + \sin^2 t + \sin^2 t + 2\cos t\sin t + \cos^2 t}\\ &= e^t \cdot \sqrt{2\cos^2 t + 2\sin^2 t}\\ &= e^t \cdot \sqrt{2}.\end{aligned}$$

$$\therefore \quad s(t) = \int_0^t \|\vec{c}\,'(\tau)\|\,d\tau = \int_0^t \sqrt{2}e^\tau\,d\tau = \sqrt{2}\cdot(e^t - 1),$$

where $t \in [0, 1]$.
Since $s = \sqrt{2}(e^t - 1)$, we can solve for t to get $t = \ln\left(\frac{s}{\sqrt{2}} + 1\right)$. Then the parametrization of $\vec{c}$ by the arc-length function is given by

$$\vec{c}(s) = \left(\left(\frac{s}{\sqrt{2}} + 1\right)\cos\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right), \left(\frac{s}{\sqrt{2}}+1\right)\sin\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)\right),$$

where $s \in [0, \sqrt{2}(e-1)]$ (here we are using the fact that $e^t = \frac{s}{\sqrt{2}} + 1$).

Definition. Let $\vec{c}$ be a curve which is the image of a smooth C^2 path in $\mathbb{R}^2$ or $\mathbb{R}^3$ parametrized by its arc-length s. The curvature $\kappa(s)$ of $\vec{c}$ at a point $\vec{c}(s)$ is given by

$$\kappa(s) = \left\| \frac{d\vec{T}(s)}{ds} \right\|,$$

where $\vec{T}(s) = \dfrac{\vec{c}\,'(s)}{\|\vec{c}\,'(s)\|}$ is the tangent unit vector.

Using the chain rule, we have the following formula for the curvature in terms of t:

$$\kappa(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{c}\,'(t)\|}.$$

Examples: (i) The curvature of a line parametrized by $\vec{c}(t) = (at, bt)$ (where $t \in \mathbb{R}$ and at least one of a or b is nonzero) is always 0:

$$\vec{T}(t) = \frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|} = \frac{(a,b)}{\sqrt{a^2+b^2}}, \text{ and so}$$

$\vec{T}\,'(t) = 0$ since $\vec{T}(t)$ is constant. Thus $\|\vec{T}\,'(t)\| = 0$ and so $\kappa(t) = 0$.

(ii) We will compute the curvature of the parabola $y = x^2$: Parametrize the parabola by $\vec{c}(t) = (t, t^2)$. Then

$$\vec{c}\,'(t) = (1, 2t), \text{ and } \|\vec{c}\,'(t)\| = \sqrt{1+4t^2}.$$

$$\therefore \vec{T}(t) = \left(\frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}} \right).$$

$$\Rightarrow \vec{T}\,'(t) = \left(\frac{d}{dt}(1+4t^2)^{-1/2}, \frac{d}{dt}\left[(2t)(1+4t^2)^{-1/2}\right] \right)$$

$$= \left(-\frac{1}{2} \cdot \frac{8t}{(1+4t^2)^{3/2}}, 2t \cdot -\frac{1}{2} \cdot \frac{8t}{(1+4t^2)^{3/2}} + 2 \cdot \frac{1}{(1+4t^2)^{1/2}} \right)$$

$$= \left(-\frac{4t}{(1+4t^2)^{3/2}}, \frac{-8t^2 + 2(1+4t^2)}{(1+4t^2)^{3/2}} \right)$$

$$= \left(-\frac{4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}} \right).$$

So $$\|\vec{T}'(t)\| = \sqrt{\frac{16t^2+4}{(1+4t^2)^3}}$$

$$= \sqrt{\frac{4(1+4t^2)}{(1+4t^2)^{3}}}$$

$$= \sqrt{\frac{4}{(1+4t^2)^2}}$$

$$= \frac{2}{1+4t^2}.$$

Now we can compute the curvature:

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{c}'(t)\|}$$

$$= \frac{2}{1+4t^2} \cdot \frac{1}{(1+4t^2)^{1/2}}$$

$$= \frac{2}{(1+4t^2)^{3/2}}.$$

Note that the curvature at the origin $c(0) = (0,0)$ is given by $\kappa(0) = 2$, and that $\kappa(t) \to 0$ as $t \to \pm\infty$. So the parabola $y = x^2$ is curved the most at the vertex $(0,0)$.

Exercise: (3.4, #25). Find the point where the graph of the function $y = \ln x$ has maximal curvature using the formula

$$\kappa(x) = \frac{|f''(x)|}{(1+(f'(x))^2)^{3/2}}$$

for the curvature of the graph of a C^2 function.

Solution: Since $y = \ln x$, we have $y' = \frac{1}{x}$ and $y'' = -\frac{1}{x^2}$. Using the formula above, we have

$$\kappa(x) = \frac{\left|-\frac{1}{x^2}\right|}{\left(1+\left(\frac{1}{x}\right)^2\right)^{3/2}} = \frac{1/x^2}{(1+1/x^2)^{3/2}}$$

$$= \frac{1/x^2}{\left(\frac{x^2+1}{x^2}\right)^{3/2}}$$

$$= \frac{1/x^2}{\frac{(x^2+1)^{3/2}}{(x^2)^{3/2}}}$$

$$= \frac{1/x^2}{\frac{(x^2+1)^{3/2}}{x^3}}$$

$$= \frac{1}{x^2} \cdot \frac{x^3}{(x^2+1)^{3/2}}$$

$$= \frac{x}{(x^2+1)^{3/2}}.$$

Since we want to find the maximum value of $\kappa(x)$, we take its derivative:

$$\kappa'(x) = \frac{d}{dx}\left(x \cdot (x^2+1)^{-3/2}\right)$$

$$= (x^2+1)^{-3/2} + x \cdot -\frac{3}{2}(x^2+1)^{-5/2} \cdot 2x$$

$$= \frac{1}{(x^2+1)^{3/2}} - \frac{3x^2}{(x^2+1)^{5/2}}$$

$$= \frac{x^2+1-3x^2}{(x^2+1)^{5/2}}$$

$$= \frac{1-2x^2}{(x^2+1)^{5/2}}.$$

We want $\kappa'(x) = 0$; this holds for $x = \pm 1/\sqrt{2}$. But $\ln x$ is only defined for positive x, and so $x = 1/\sqrt{2}$ is the only possible extreme point.

We want to check that $x = 1/\sqrt{2}$ is indeed the point where the maximal curvature occurs. Indeed, if $0 < x < 1/\sqrt{2}$, then

$$0 < x^2 < \frac{1}{2} \Rightarrow 0 > -2x^2 > -1$$
$$\Rightarrow 1 > 1-2x^2 > 0,$$

in particular $1-2x^2 > 0$ and so $\kappa'(x) > 0$ (since the denominator is always positive). Similarly $x > 1/\sqrt{2}$ implies $\kappa'(x) < 0$. So $1/\sqrt{2}$ must be the point where the maximum curvature occurs. The value at that point is

$$\kappa(1/\sqrt{2}) = \frac{1/\sqrt{2}}{(3/2)^{3/2}} \approx 0.3849.$$