Math 2A03 Tutorial #7 (Oct. 27 – 28.)

Definition. Let $\vec{c} : [a,b] \to \mathbb{R}^2$ be a C^1 path and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(\vec{c}(t))$ is continuous on $[a,b]$. The *path integral* of f along $\vec{c}$ is

$$\int_{\vec{c}} f \, ds = \int_a^b f(\vec{c}(t)) \cdot \|\vec{c}\,'(t)\| \, dt.$$

Exercise: Compute $\int_{\vec{c}} f \, ds$ for the following functions f and paths $\vec{c}$:

(a) $f(x,y) = 2x - y$; $\vec{c}(t) = (e^t + 1, e^t - 2)$, $0 \leq t \leq \ln 2$.

(b) $f(x,y,z) = y - z^2$; $\vec{c}(t) = (t^2, \ln t, 2t)$, $1 \leq t \leq 4$.

(c) $f(x,y,z) = (x^2 + y^2 + z^2)^{-1}$; $\vec{c}(t) = (t,t,t)$, $1 \leq t < \infty$.

Solution: (a) $f(\vec{c}(t)) = f(e^t + 1, e^t - 2) = 2(e^t + 1) - (e^t - 2)$

$= 2e^t + 2 - e^t + 2 = e^t + 4.$

$\vec{c}\,'(t) = (e^t, e^t)$ so

$\|\vec{c}\,'(t)\| = \sqrt{(e^t)^2 + (e^t)^2} = \sqrt{2e^{2t}} = \sqrt{2} \cdot e^t.$

Now we can compute $\int_{\vec{c}} f \, ds$:

$$\int_c f \, ds = \int_0^{\ln 2} (e^t + 4)(\sqrt{2} \cdot e^t) \, dt$$

$$= \sqrt{2} \cdot \int_0^{\ln 2} e^{2t} + 4e^t \, dt$$

$$= \sqrt{2} \cdot \left(\frac{e^{2t}}{2} + 4e^t \right) \Big|_0^{\ln 2}$$

$$= \sqrt{2} \cdot \left[\frac{e^{2\ln 2}}{2} + 4e^{\ln 2} - \left(\frac{e^0}{2} + 4e^0 \right) \right]$$

$$= \sqrt{2} \cdot \left(2 + 8 - \frac{1}{2} - 4 \right)$$

$$= \sqrt{2} \cdot \left(\frac{12}{2} - \frac{1}{2} \right)$$

$$= \sqrt{2} \cdot \frac{11}{2}.$$

(b) $f(\vec{c}(t)) = f(t^2, \ln t, 2t)$
$= \ln t - 4t^2.$

$\vec{c}\,'(t) = (2t, t^{-1}, 2)$, so

$\|\vec{c}\,'(t)\| = \sqrt{4t^2 + t^{-2} + 4} = \sqrt{(2t + t^{-1})^2} = 2t + t^{-1}.$

(Note that we can actually take the square root here, since $t \in [1,4]$ and so t is always non-negative.)

Thus

$$\int_{\vec{c}} f\, ds = \int_1^4 (\ln t - 4t^2)(2t + t^{-1})\, dt$$

$$= \int_1^4 (2t \ln t + t^{-1} \ln t - 8t^3 - 4t)\, dt$$

$$= 2\int_1^4 t \ln t\, dt + \int_1^4 \frac{\ln t}{t}\, dt - 8\int_1^4 t^3\, dt - 4\int_1^4 t\, dt.$$

$$-8 \cdot \int_1^4 t^3\, dt = -8 \cdot \frac{t^4}{4}\Big|_1^4 = -2t^4\Big|_1^4 = -512 + 2 = -510.$$

$$-4 \cdot \int_1^4 t\, dt = -4 \cdot \frac{t^2}{2}\Big|_1^4 = -2t^2\Big|_1^4 = -32 + 2 = -30.$$

Using integration by parts with $u = \ln t$, $dv = t\,dt$ (so $du = \frac{1}{t}dt$, $v = \frac{t^2}{2}$), we get

$$\int_1^4 t \ln t\, dt = \frac{1}{2}t^2 \ln t\Big|_1^4 - \int_1^4 \frac{t}{2}\, dt$$

$$= \left(8 \ln 4 - \frac{1}{2}\ln 1\right) - \frac{t^2}{4}\Big|_1^4$$

$$= 8\ln 4 - \left(4 - \frac{1}{4}\right) = 8 \ln 4 - \frac{15}{4}.$$

Using the substitution $u = \ln t$, we get

$$\int_1^4 \frac{\ln t}{t}\, dt = \int_0^{\ln 4} u\, du = \frac{u^2}{2}\Big|_0^{\ln 4} = \frac{(\ln 4)^2}{2}.$$

Combining all of the above integrals, we get

$$\int_{\vec{c}} f\, ds = 2\left(8 \ln 4 - \frac{15}{4}\right) + \frac{(\ln 4)^2}{2}$$

$$-510 \quad -30$$

$$= 16 \ln 4 - \frac{15}{2} - 540 + \frac{(\ln 4)^2}{2}$$

$$\approx -524.36.$$

(c) $f(\vec{c}(t)) = f(t,t,t) = (t^2+t^2+t^2)^{-1} = (3t^2)^{-1}$.

$\vec{c}\,'(t) = (1,1,1)$, so we get

$\|\vec{c}\,'(t)\| = \sqrt{3}$. Since $1 \le t < \infty$, we have to look at $1 \le t \le b$ and the compute the limit as $b \to \infty$: Let $\vec{c}_b(t) = (t,t,t)$ for $1 \le t \le b$. Then

$$\int_{\vec{c}_b} f\, ds = \int_1^b \frac{t^{-2}}{3} \cdot \sqrt{3}\, dt$$

$$= \frac{\sqrt{3}}{3} \cdot -t^{-1}\Big|_1^b = -\frac{\sqrt{3}}{3}\left(\frac{1}{b} - 1\right) = \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3b}.$$

Thus we can compute the limit as $b \to \infty$:

$$\int_{\vec{c}} f\, ds = \lim_{b\to\infty} \int_{\vec{c}_b} f\, ds$$

$$= \lim_{b\to\infty}\left(\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3b}\right)$$

$$= \frac{\sqrt{3}}{3} - \lim_{b\to\infty} \frac{\sqrt{3}}{3} \cdot \frac{1}{b} = \frac{\sqrt{3}}{3}.$$

<u>Definition</u>. Let $\vec{c}:[a,b]\to\mathbb{R}^2$ (or $\mathbb{R}^3$) be a C^1 path, and let $\vec{F}:\mathbb{R}^2\to\mathbb{R}^2$ (or $\mathbb{R}^3\to\mathbb{R}^3$) be a vector field such that $\vec{F}(\vec{c}(t))$ is continuous on $[a,b]$. The <u>path integral</u> (or <u>line integral</u>) of $\vec{F}$ along the path $\vec{c}$ is

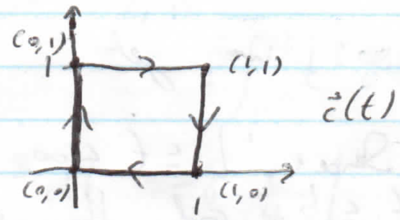
$$\int_{\vec{c}} \vec{F}\cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t))\cdot \vec{c}\,'(t)\,dt.$$

<u>Exercise</u>: Compute $\int_{\vec{c}} \vec{F}\cdot d\vec{s}$:

(a) $\vec{F}(x,y) = (e^{x+y}, -1)$; $\vec{c}$ is the boundary of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$, oriented clockwise.

(b) $\vec{F}(x,y) = (2xy, e^y)$; $\vec{c}(t) = (4t^3, t^2)$, $t\in[0,1]$.

<u>Solution</u>: (a) $\vec{c}(t)$ looks like



We need to parametrize each of the four segments individually:

Parametrize the segment from $(0,0)$ to $(0,1)$ by $\vec{c}_1(t) = (0, t)$, $0\le t\le 1$;

Parametrize the segment from $(0,1)$ to $(1,1)$ by $\vec{c}_2(t) = (t, 1)$, $0\le t\le 1$;

Parametrize the segment from $(1,1)$ to $(1,0)$ by $\vec{c}_3(t) = (1, 1-t)$, $0\le t\le 1$;

Parametrize the segment from $(1,0)$ to $(0,0)$ by $\vec{c}_4(t) = (1-t, 0)$, $0\le t\le 1$.

$\vec{F}(\vec{c}_1(t)) = (e^t, -1)$; $\quad \vec{c}_1{}'(t) = (0, 1)$.
$\vec{F}(\vec{c}_2(t)) = (e^{t+1}, -1)$; $\quad \vec{c}_2{}'(t) = (1, 0)$.
$\vec{F}(\vec{c}_3(t)) = (e^{2-t}, -1)$; $\quad \vec{c}_3{}'(t) = (0, -1)$.
$\vec{F}(\vec{c}_4(t)) = (e^{1-t}, -1)$; $\quad \vec{c}_4{}'(t) = (-1, 0)$.

Now we can compute the line integral:

$$\int_{\vec{c}} (e^{x+y}, -1)\cdot d\vec{s} = \int_{\vec{c}_1} (e^{x+y}, -1)\cdot d\vec{s} + \int_{\vec{c}_2} (e^{x+y}, -1)\cdot d\vec{s} + \int_{\vec{c}_3} (e^{x+y}, -1)\cdot d\vec{s} + \int_{\vec{c}_4} (e^{x+y}, -1)\cdot d\vec{s}$$

$$= \int_0^1 (e^t, -1)\cdot(0,1)\,dt + \int_0^1 (e^{t+1}, -1)\cdot(1,0)\,dt + \int_0^1 (e^{2-t}, -1)\cdot(0,-1)\,dt + \int_0^1 (e^{1-t}, -1)\cdot(-1,0)\,dt$$

$$= \int_0^1 -1\,dt + \int_0^1 e^{t+1}\,dt + \int_0^1 1\,dt + \int_0^1 -e^{1-t}\,dt$$

$$= -1 + \cancel{}\, e^{t+1}\Big|_0^1 + 1 + e^{1-t}\Big|_0^1 = e^2 - 2e + 1.$$

(b) $\vec{F}(\vec{c}(t)) = \vec{F}(4t^3, t^2) = (8t^5, e^{t^2})$.
$\vec{c}\,'(t) = (12t^2, 2t)$. We compute the line integral:

$$\int_{\vec{c}} (2xy, e^y)\cdot d\vec{s} = \int_0^1 (8t^5, e^{t^2})\cdot(12t^2, 2t)\,dt$$

$$= \int_0^1 96t^7 + 2te^{t^2}\,dt.$$

$$\int_0^1 2te^{t^2}\,dt \overset{u=t^2}{=} \int_0^1 e^u\,du = e^u\Big|_0^1 = e-1,$$

So

$$\int_{\vec{c}} (2xy, e^y)\cdot d\vec{s} = \frac{96t^8}{8}\Big|_0^1 + (e-1)$$

$$= 12 + e - 1 = 11 + e.$$

Exercise: (5.3 #21) Compute

$$\int_{\vec{c}} \frac{(y\,dx + x\,dy)}{(x^2+y^2)},$$

where $\vec{c}$ is the circle centred at the origin of radius 2 oriented counterclockwise.

Solution: (Recall that, for real-valued functions F_1 and F_2,

$$\int_{\vec{c}} F_1\,dx + F_2\,dy = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt}\right) dt$$

where $\vec{c}(t) : [a,b] \to \mathbb{R}^2$ is a C^1 path with components $\vec{c}(t) = (x(t), y(t))$.)

Parametrize the given circle by

$$\vec{c}(t) = (2\cos t, 2\sin t), \quad t \in [0, 2\pi].$$

Since $x^2+y^2 = 4$ on $\vec{c}$, we have

$$\int_{\vec{c}} \frac{y\,dx + x\,dy}{x^2+y^2} = \frac{1}{4}\int_0^{2\pi} (2\sin t \cdot -2\sin t + 2\cos t \cdot 2\cos t)\,dt$$

$$= \frac{1}{4}\int_0^{2\pi} 4(\cos^2 t - \sin^2 t)\,dt$$

$$= \int_0^{2\pi} \cos^2 t - \sin^2 t\,dt$$

$$= \int_0^{2\pi} \cos 2t\,dt$$

$$= \frac{1}{2}\sin 2t \Big|_0^{2\pi} = 0.$$