MATH 3TP3 Assignment #10
Due: Friday, November 30, in class

In this assignment, I ask you to prove that various things are or are not computable (/computably enumerable). For this, informal arguments about the existence of algorithms, in the intuitive sense, will suffice. (If one wanted to be rigorous, one would argue with respect to formally described Turing complete systems, e.g. showing (non-)existence of register machine programs.)

Suppose we have fixed a Gödel numbering for an alphabet which contains the alphabet of TNT and also the dash symbol "-".

Define the function Dashify : $\mathbb{N} \to \mathbb{N}$ by

$$\text{Dashify}(n) = \ulcorner \text{-}^n \urcorner,$$

where "-^n" is the string consisting of n dashes.

Show that Dashify **is a (total) computable function, by describing informally an algorithm to calculate it.**

Since string manipulations and finding free variables correspond to computable operations on Gödel numbers, it follows that for any TNT-wff with one free variable $\phi(x)$, the function $\text{DashSub}_\phi : \mathbb{N} \to \mathbb{N}$ defined by

$$\text{DashSub}_\phi(n) = \ulcorner \phi(\overline{\ulcorner \text{-}^n \urcorner}) \urcorner$$

is computable. You should ponder this, but you don't need to write anything.

Recall that we showed, as part of our discussion of the Halting problem, that there exists a set H which is c.e. but not computable.

Consider Dashify(H), the image of H under the dashification map Dashify. **Show that** Dashify(H) **is c.e. but not computable.**

Deduce that there is a Post formal system S for which -^n is an S-theorem iff $n \in H$.

Without loss of generality, assume that we can extend our Gödel numbering to include the alphabet of S.

Now

$$n \in H \iff \mathbb{N} \vdash \text{Theorem}_S(\overline{\ulcorner \text{-}^n \urcorner}); \tag{1}$$

as we saw (/will see) in our discussion of Gödel's Second Incompletness Theorem, it follows:

$$n \in H \iff \text{TNT} \vdash \text{Theorem}_S(\overline{\ulcorner \text{-}^n \urcorner}). \tag{2}$$

By considering DashSub_ϕ for an appropriate ϕ, **show that**

$$\{\ulcorner \text{Theorem}_S(\overline{\ulcorner \text{-}^n \urcorner}) \urcorner \mid n \notin H\}$$

is *not* c.e.

Deduce from this and (2) that the set of Gödel numbers of TNT-sentences which are not TNT-theorems is not c.e.

Conclude that there does not exist an "anti-TNT", a formal system which proves precisely those TNT-sentences which TNT does not prove.