MATH 3TP3 Assignment #2

Due: Friday, 28th of September, in class

1. Consider the following formal system:

   - Alphabet: $\{\texttt{-}, \texttt{!}\}$;
   - Axioms: $\{\texttt{-!--}\}$;
   - Production rules:

     (I) $\texttt{-!}x \mapsto \texttt{-!-}x$

     (II) $x\texttt{!}y \mapsto x\texttt{-!-}y$

     (III) $x\texttt{!}y \mapsto y\texttt{!}x$

   (a) Find derivations in the system of `--!----` and `---!-`.

   (b) Consider the language consisting of strings of the form $x\texttt{!}y$ where x and y are (non-empty) strings of hyphens.

   Give an interpretation for this language with respect to which the system is consistent and complete. Briefly explain your reasoning.

2. We have seen that the `pq`-system captures addition of positive integers $n+m = k$, and the `tq`-system captures multiplication $nm = k$. Devise a formal system (with finitely many axioms and production rules) which captures exponentiation $n^m = k$.

   Argue informally that your system is consistent and complete with respect to the intended interpretation.

   Hint: we want a formal system on an alphabet including $\texttt{e}, \texttt{q}$ such that $\texttt{-}^n\texttt{e}\texttt{-}^m\texttt{q}\texttt{-}^k$ is a theorem iff $n^m = k$. Your first thought might be to copy the pattern of the previous systems and have a production rule "given $x\texttt{e}y\texttt{q}z$, produce $x\texttt{e}y\texttt{-q}zz\ldots z$ with as many copies of z as x has hyphens" - but sadly this is **not** a typographical rule of the kind allowed in our formal systems. So you need to find a way around this. You may find it helpful to make use of systems we have already developed, as we did when we captured compositeness.

3. Determine which of the following strings are well formed. For those that are, produce their parse (formation) trees.

   (a) $\langle P \wedge P' \wedge P \rangle$

(b) $\langle P \wedge \langle P' \wedge P \rangle\rangle$

(c) $\langle P \wedge \sim\sim \langle Q \vee R \rangle\rangle$

(d) $\langle\langle\langle P \supset R \rangle\rangle \vee R \rangle$

4. For each of the following wff's, determine whether it is true or false according to an interpretation under which P and Q are true and R is false:

   (a) $\langle P \supset \langle Q \supset P \rangle\rangle$

   (b) $\langle\langle R \wedge Q \rangle \supset \sim Q \rangle$

   (c) $\langle\langle P \supset \sim P \rangle \wedge \langle \sim P \supset P \rangle\rangle$

   (d) $\langle\langle\langle\langle P \supset Q \rangle \supset R \rangle \supset \langle\langle R \supset P \rangle \supset Q \rangle\rangle \supset \langle\langle Q \supset R \rangle \supset P \rangle\rangle$

Bonus Question: So far, numbers have been represented in our formal systems in a rather primitive way, as rows of dashes. Can you devise a formal system to capture addition and multiplication of, say, binary numerals in place of hyphen strings? The question below guides you towards my own solution to this puzzle, but you might like to try to find your own solution before reading it!

(a) First, we capture successorship, $m = n + 1$, with binary numerals:

- Alphabet: $\{\texttt{0}, \texttt{1}, \texttt{S}, \texttt{C}\}$
- Axiom: `1S0`
- Production rules:
  - (I) $x\texttt{S}y \mapsto x\texttt{CS}y\texttt{C}$
  - (II) $x\texttt{0C}y \mapsto x\texttt{1}y$
  - (III) $x\texttt{1C}y \mapsto x\texttt{C0}y$
  - (IV) $x \mapsto \texttt{0}x$
  - (V) $x\texttt{S}y \mapsto x\texttt{S0}y$

Derive `10S1` and `11S10`. Convince yourself that if x and y are binary numerals, $x\texttt{S}y$ is a theorem precisely when $x = y + 1$.

(b) Using the above system, or otherwise, find a formal system with finitely many axioms and production rules in which whenever x, y and z are binary numerals, $x\texttt{p}y\texttt{q}z$ is a theorem iff $x + y = z$.

(c) Do the same for multiplication. Briefly indicate how you would revise the above system to work with base ten rather than base two.