MATH 3TP3 Assignment #1 Solutions

1. *Find a derivation in the* `MIU`*-system of the string* `MIUI`*.*

$$\texttt{MI}, \texttt{MII}, \texttt{MIIII}, \texttt{MIIIIIIII}, \texttt{MIIIIIIIIU},$$
$$\texttt{MIIIIIUU}, \texttt{MIIIII}, \texttt{MIUI}$$

2. *Write down* **all** *the derivations (including the "stupid" ones) in the* `MIU`*-system of length (number of lines) at most 3 (Hint: there's only 1 of length 1, there are 3 of length 2, and there are 12 of length 3).*

(MI);		
(MI, MI);	(MI, MIU);	(MI, MII);
(MI, MI, MI);	(MI, MI, MIU);	(MI, MI, MII);
(MI, MIU, MIU);	(MI, MIU, MI);	(MI, MIU, MIIU);
(MI, MIU, MII);	(MI, MII, MI);	(MI, MII, MII);
(MI, MII, MIIII);	(MI, MII, MIIU);	(MI, MII, MIU)

3. *Consider the system* `MIU+`*, the variant of the* `MIU`*-system obtained by adding as a fifth production rule: given* `MU`x *and* `MU`y*, produce* `MU`xy*. Find a derivation of* `MU` *in this system.*

$$\texttt{MI}, \texttt{MII}, \texttt{MIIII}, \texttt{MUI}, \texttt{MIIIIIIII},$$
$$\texttt{MUIIIII}, \texttt{MUIIIIII}, \texttt{MUUIII}, \texttt{MUUU}, \texttt{MU}$$

4. (a) *Let x be a string of Is of length 2^n for some integer $n \geq 0$. Show that* `M`x *is an* `MIU`*-theorem.*

   We can prove this by induction on n.

   For $n = 0$, $2^n = 1$ and $\texttt{MI}^{2^n} = \texttt{MI}$, which is an axiom, and hence a theorem.

   Assume that $\texttt{MI}^{2^n}$ is a theorem. Then by applying Rule (II), we see that $\texttt{MI}^{2^n}\texttt{I}^{2^n} = \texttt{MI}^{2^{n+1}}$ is also a theorem.

(b) *Let x be a string of Is whose length l is not a multiple of 3. Show that the string $\mathtt{M}x$ is an* `MIU`*-theorem. Hint: first show that there is some number n such that 2^n is congruent to l modulo 3 and $2^n \geq l$.*

First, we find an n as in the hint. Let m be such that $2^m \geq l$. Since 2 is prime, 2^m is not divisible by 3, so is congruent to either 1 or 2 mod 3. So $2^{m+1} = 2 \cdot 2^m$ is congruent respectively to 2 or 1 mod 3. So since l is not divisible by 3, one of 2^m and 2^{m+1} is congruent to l mod 3. Let $n = m$ or $n = m + 1$ accordingly. So $l = 2^n - 3k$ for some $k \geq 0$.

Now from part (a) we have that $\mathtt{MI}^{2^n}$ is a theorem. Note that applying Rules (I), (III) then (IV) to a string $\mathtt{MI}^{3+m}$ produces $\mathtt{MI}^m$. So applying this succession of rules k times to $\mathtt{MI}^{2^n}$ produces $\mathtt{MI}^l$. So $\mathtt{MI}^l$ is a theorem.

(c) *Let x be a string over the alphabet $\{\mathtt{I}, \mathtt{U}\}$ such that the number of occurrences of the symbol* `I` *in x is* ***not*** *a multiple of 3. Show that the string $\mathtt{M}x$ is an* `MIU`*-theorem.*

Let l be the number of occurrences of `I` in x and k the number of occurrences of `U` in x.

Let $l + 3k$ is congruent to l modulo 3, so is not divisible by 3. Thus by part (b), the string $\mathtt{MI}^{l+3k}$ is a theorem. With k judicious applications of Rule (III), we can produce Mx from it.

(d) *From this and the solution to the* `MU`*-puzzle given in lectures, you can conclude that there is a decision procedure for* `MIU`*-theoremhood. Explain briefly how to decide, given an* `MIU`*-string S, whether or not S is an* `MIU`*-theorem.*

We saw in class that if $\mathtt{M}x$ is a theorem then the number of occurrences of `I` in x cannot be a multiple of 3. From part c) we know that the converse holds and so $\mathtt{M}x$ is a theorem if and only if the number of occurrences of `I` in x is not divisible by 3.

We also saw in class that every theorem is of the form Mx, where x is a string in the alphabet $\{\mathtt{I}, \mathtt{U}\}$.

Thus, to recognize whether a string is an `MIU`-theorem, we need only check that it starts with `M`, has no other occurrences of `M` and that the number of occurrences of `I` in the string is not divisible by 3.

BONUS: *Consider the function* C *from natural numbers to natural numbers which takes* n *to* $n/2$ *if* n *is even, and to* $3n+1$ *if* n *is odd. The Collatz conjecture (currently unsolved) states that for every* $n > 0$*, if you repeatedly apply this function you will eventually get 1 (i.e.* $C(n) = 1$ *or* $C(C(n)) = 1$ *or* $C(C(C(n))) = 1$ *or* $\ldots$*).*

Find a formal system with an alphabet including C *such that the Collatz conjecture is true if and only if every string consisting entirely of* C*s is a theorem of the system.*

Here's my solution, using a three-character alphabet. I'd be interested to see a natural way of doing it with only two.

- Alphabet: $\{\texttt{C}, \texttt{I}, \texttt{S}\}$;
- Axioms: $\{\texttt{S}\}$;
- Production rules:

  (I) $x\texttt{S}y \mapsto x\texttt{CSI}y$

  (II) $x\texttt{S}y \mapsto xy$

  (III) $x\texttt{C}yy \mapsto x\texttt{C}y$

  (IV) $x\texttt{C}yy\texttt{I} \mapsto x\texttt{C}yyyyyy\texttt{IIII}$

  (V) $x\texttt{CI} \mapsto x\texttt{C}$