MATH 3TP3 Assignment #6 Solutions

1. (a) False in $\mathbb{N}$.

   (b) [

   $\forall x : (x + x) = y$
   $(x + x) = y$
   $\exists y : (x + x) = y$
   $\forall x : \exists y : (x + x) = y$

   ]
   $\langle \forall x : (x + x) = y \supset \forall x : \exists y : (x + x) = y \rangle$
   [

   $\sim \forall x : \exists y : (x + x) = y$
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   $\sim \exists y : \forall x : (x + x) = y$

   ]
   $\langle \sim \forall x : \exists y : (x + x) = y \supset \sim \exists y : \forall x : (x + x) = y \rangle$
   $\langle \exists y : \forall x : (x + x) = y \supset \forall x : \exists y : (x + x) = y \rangle$

   (c) This is false in the structure of arithmetic modulo 2.

   (d) This is false in the structure obtained from $\mathbb{N}$ by setting `0` to be interpreted as 37.

   (e) This is false in the structure obtained from $\mathbb{N}$ by adjusting addition as follows: $n +' m := n + m$ if $m \neq 2$, but $n +' 2 = 73$ for all n.

2. (a) TNT' can prove $\forall x : (x \cdot S0) = (0 + x)$, so if it could prove $\forall x : (x \cdot S0) = x$ then it would also prove $\forall x : (0 + x) = x$. But as was mentioned in lectures, it can't prove that! So there must be a structure which satisfies the axioms of TNT' but in which $\forall x : (x \cdot S0) = x$ is false.

   How can we find such a structure? There are a few possibilities, here's a cute one: we add a second copy of $\mathbb{N}$ to $\mathbb{N}$, and have addition and multiplication fold everything down to one copy or the other - folding to the left with addition, and to the right with multiplication.

More explicitly: we add new elements $0', 1', 2', \ldots$ to $\mathbb{N}$, extend successor to the new elements by defining $S(n') := (n+1)'$, and extend addition by taking the primedness of the result to be the primedness of the first summand, and for multiplication take it from the second multiplicand. In other words:

$$n + m' = n + m;\ n' + m' = n' + m = (n+m)',$$

and

$$n' \cdot m = n \cdot m;\ n' \cdot m' = n \cdot m' = (n \cdot m)'.$$

One can see that the axioms of TNT' hold for this structure. But e.g. $(0' \cdot 0) = 0$, so $\forall x : (x \cdot S0) = x$ is false for this structure; similarly $(0 + 0') = 0$ so $\forall x : (0 + x) = x$ is false in the structure.

(Remark: it's common to work with a stronger system than TNT', known as Robinson arithmetic and denoted by Q, which adds to TNT' the axiom $\forall x : \langle x = 0 \vee \exists y : Sy = x \rangle$ (which is a TNT-theorem). Our structure does not satisfy this last axiom. However, $\forall x : (x \cdot S0) = x$ and $\forall x : (0 + x) = x$ are also not theorems of Q. There's a reasonably pleasant example of a structure demonstrating this on page 7 of Peter Smith's *Gödel Without Tears* (linked to from the the course website), obtained from $\mathbb{N}$ by adding two extra elements.)

(b) $(SS0 \cdot SS0) = ((SS0 \cdot S0) + SS0)$
$(SS0 \cdot S0) = ((SS0 \cdot 0) + SS0)$
$(SS0 \cdot 0) = 0$
$((SS0 \cdot 0) + SS0) = (0 + SS0)$
$(SS0 \cdot S0) = (0 + SS0)$
$(0 + SS0) = S(0 + S0)$
$(0 + S0) = S(0 + 0)$
$(0 + 0) = 0$
$S(0 + 0) = S0$
$(0 + S0) = S0$
$S(0 + S0) = SS0$
$(0 + SS0) = SS0$
$(SS0 \cdot S0) = SS0$
$((SS0 \cdot S0) + SS0) = (SS0 + SS0)$
$(SS0 \cdot SS0) = (SS0 + SS0)$

Remark: We had to prove here $(0+SS0) = SS0$: this is a theorem of TNT', even though $\forall x : (0 + x) = x$ is not.

(c) This is false in $\mathbb{N}$.

(d) This is false in $\mathrm{Mat}_2(\mathbb{N})$.