Introduction to 3TP3: Hilbert's Programme and Gödel's Incompleteness Theorems

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September 7, 2012

Hilbert's 2nd problem

In the second of David Hilbert's famous series of problems from his 1900 address to the ICM, he set forth the problem of proving consistency of axioms for analysis:

> *"When we are engaged in investigating the foundations of a science, we must set up a system of axioms which contains an exact and complete description of the relations subsisting between the elementary ideas of that science. [...] above all I wish to designate the following as the most important among the numerous questions which can be asked with regard to the axioms: To prove that they are not contradictory, that is, that a definite number of logical steps based upon them can never lead to contradictory results. [...] I am convinced that it must be possible to find a direct proof for the compatibility of the arithmetical axioms"*

Consistency

In modern terminology:

- A *formal system* is a collection of axioms and rules of inference for making deductions from those axioms.
- A formal system is *inconsistent* if it allows us to deduce a statement S and also to deduce its negation “not S”.

Hilbert wanted a formal system in which the achievements of mathematics could be deduced, and which could satisfactorily be **proven** to be consistent.

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- Is R an element of R?
- R is an element of R iff R is not an element of R.
- Oh dear.

Avoiding Russel

1900-1930: attempts at formalisation which avoided Russel's paradox, by disallowing self-reference of the kind used to define R.

- Whitehead and Russel's *Principia Mathematica*;
- Zermelo-Frankel set theory.

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Step 2 : Consider the statements and proofs in the formal system as mathematical objects in themselves;

Step 3 : Apply purely "finitistic" mathematical reasoning to these objects to prove the mathematical statement: there does not exist a formal statement S such that there exist formal proofs of S and of "not S".

No *ignorabimus*

Hilbert 1900: "Take any definite unsolved problem, such as the question as to the irrationality of the Euler-Mascheroni constant C, or the existence of an infinite number of prime numbers of the form $2^n + 1$. However unapproachable these problems may seem to us and however helpless we stand before them, we have, nevertheless, the firm conviction that their solution must follow by a finite number of purely logical processes. [...] This conviction of the solvability of every mathematical problem is a powerful incentive to the worker. We hear within us the perpetual call: There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no *ignorabimus*."

Completeness

A formal system is *complete* if for any statement S in the system, either there is proof in the system of S, or there is a proof of "not S".

Mechanisation of Mathematics

Suppose we have a complete formal system, we believe the axioms to be true and the rules of inference to be valid, and we want to know whether or not a statement S is true.

Mechanically run through all possible proofs in the system, blindly applying the rules of inference to the axioms in all possible combinations. By completeness, eventually one of S or “not S” will be proved.

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He showed

- Gödel’s First Theorem: Assuming it is consistent, Principia Mathematica, or **any** formal system of anything like its scope, is incomplete.
- Gödel’s Second Theorem: Assuming it is consistent, such a formal system can not prove that it is consistent.