3U03 Midterm I [2015] Sample Sol$^{(n)}$

1. Count the 7-digit numbers <u>with</u> 3 consecutive equal digits by partitioning according to the <u>first</u> digit which begins such a sequence.

If it's the first there are 9 possibilities for the repeated digit, and 10^4 for the subsequent 4 digits.

If it's the 2^{nd} there are 10 possibilities for the repeated digit, 10 for each of the last 3, but only 8 for the 1^{st}, since it can't be 0 nor equal to the repeated digit.

If it's the 3^{rd} or 4^{th}, similarly we have 9 possibilities for the 1^{st} digit, 9 for the one just before the first repeated one, and 10 for the other 2.

If it's the 5^{th} we have the same possibilities except that we must exclude the 9 numbers where the first 3 digits are equal.

So answer $= 9000000 - (9\times10^4) - (10\times8\times10^3) - 2(10\times9^2\times10^2) - (10\times9^2\times10^2 - 9)$
$= 8587009$

2. There are $4\times8 = 32$ circles in the diagram.

Partition the 73 points according to "the" ~~[illegible]~~ disc in which each lands, by making some arbitrary choice of disc to choose when the point is in multiple discs, e.g. choosing the lowest-then-leftmost.

(see overleaf for the pretty partition of the square this yields)

This gives a map from the 73 points to the 32 discs.

Since $73 > 32\times2$

by the "Packed Pigeonhole Principle", some 3 points are mapped to the same disc which implies that they are contained in that disc of radius one.

3. Let $|S| = 2n+1$.

Partition S into subsets S_1 and S_2 with $|S_1| = n$ and $|S_2| = n+1$

$S = S_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$.

If $X \subseteq S$ is a subset of size n we may put in correspondence to it the pair of sets $X_1 := X \cap S_1 \subseteq S_1$
$X_2 := X \cap S_2 \subseteq S_2$

$$X \sim (X \cap S_1,\ X \cap S_2)$$

This is a 1-1 correspondence between the subsets of X of size n and the pairs of subsets of S_1 and S_2 respectively with sizes summing to n.

~~[illegible]~~ Partitioning the latter according to the size of the first subset and applying the addition principle, we obtain

$$\binom{2n+1}{n} = \#(n\text{-subsets of } X)$$
$$= \#(\text{pairs as above})$$
$$= \sum_{r=0}^{n} \#(r\text{-subsets of } X_1)\,\#((n+1-r)\text{-subsets of } X_2)$$
$$= \sum_{r=0}^{n} \binom{n}{r}\binom{n+1}{n+1-r} = \sum_{r=0}^{n} \binom{n}{r}\binom{n+1}{r} \quad \square$$

4. By Ramsey's Theorem, for some N, $K_N \to K_5, K_5$.

Suppose the sequence is of length $n \geq N$.

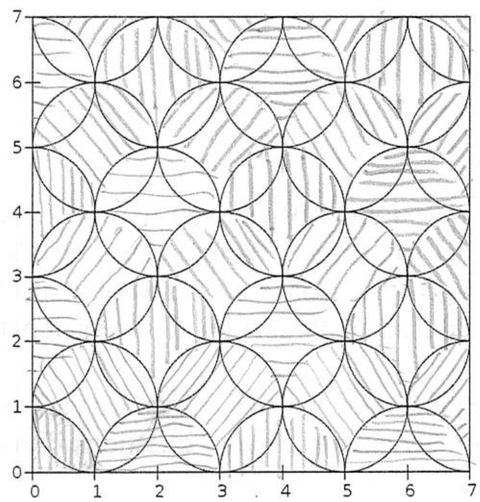
Consider the graph with points corresponding to the elements in the sequence connected by a blue edge if the elements are prime-separated, red else.

Since $K_n \to K_5, K_5$, there is a red K_5 or a blue K_5 i.e. either (i) some 5 of the integers are all pairwise prime-sep^d
or (ii) " " " " " " " not prime-sep^d

4. cont^d So if (ii) is false, i.e. for any 5 some pair is prime-sep^d, then (i) is true. $\square$

1. [5] How many 7-digit numbers (the integers between 1000000 and 9999999) have no three consecutive digits equal?

2. [5] Consider the square $[0,7] \times [0,7]$. and (partial) discs of radius one with centres (i, j) in the square where i and j are integers with $i+j$ even.



tesselates

Suppose 73 points are chosen within this square. By considering the above diagram, or otherwise, show that some disc of radius one contains at least three of the points.

3. [5] Prove that for any $n \geq 0$,

$$\sum_{r=0}^{n} \binom{n}{r} \binom{n+1}{r} = \binom{2n+1}{n}.$$

4. [5] Say a pair of integers (a, b) is <u>prime-separated</u> if their difference is a prime number (i.e. $|a-b|$ is prime).

   Prove that for sufficiently large n, if a sequence of n integers has the property that for **any** 5 of the integers, **some** pair amongst the 5 is prime-separated, then for **some** 5 of the integers, **every** pair amongst the 5 is prime-separated.