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Some Harder Games and How to Make Them Easier

Our life is frittered away by detail . . . . Simplify, simplify.
Henry David Thoreau, *Walden*.

Nim? Yes, yes, yes, let's nim with all my heart.
John Byrom, *The Nimmers*, 27

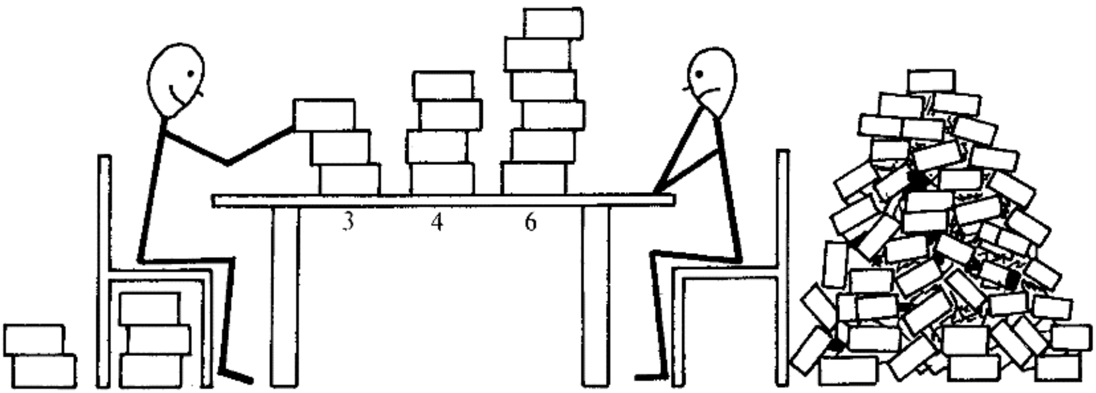

Figure 1. A Well Advanced Game of Poker-Nim.

Poker-Nim

This game is played with heaps of Poker-chips. Just as in ordinary Nim, either player may reduce the size of any heap by removing some of the chips. But now we allow a player the alternative move of increasing the size of some heap by adding to it some of the chips he acquired in earlier moves. The two kinds of move are the only ones allowed.

Let's suppose there are three heaps, of sizes 3, 4, 6 as in Fig. 1, and that the game has been going on for some time, so that both players have accumulated substantial reserves of chips. It's Left's turn to move, and he moves to 2, 4, 6 since he remembers from Chapter 2 that this is a good move in ordinary Nim. But now Right adds 50 chips to the 4 heap, making the position 2, 54, 6, which is well beyond those discussed in Chapter 2.

This seems somewhat disconcerting, especially since Right has plenty more chips at his disposal, and doesn't seem too scared of using them to complicate the position. What does Left do? After a moment's thought, he just removes the 50 chips Right has just added and waits for Right's reply. If Right adds 1000 chips to one of the heaps, Left will remove them and restore the position to 2, 4, 6 again. Sooner or later, Right must reduce one of the three heaps (since otherwise he'll run out of chips no matter how many he has), and then Left can reply with the appropriate Nim-move.

So whoever can win a position in ordinary Nim can still win in Poker-Nim, no matter how many chips his opponent has accumulated. He replies to the opponent's reducing moves just as he would in ordinary Nim, and reverses the effect of any increasing moves by using a reducing move to restore the heap to the same size again. The new moves in Poker-Nim can only postpone defeat, not avoid it indefinitely. Since the effect of any of the new moves can be immediately reversed by the other player, we call them **reversible moves**..

Northcott's Game

The same sort of thing happens in other games, often in better disguise. Northcott's game is played on a checkerboard which has one black piece and one white piece on each row, as in Fig. 2. You may move any piece of your own color to another empty square in the same row, provided you do not jump over your opponent's piece in that row. If you can't move (because all your pieces are trapped at the side of the board by your opponent's), you lose.

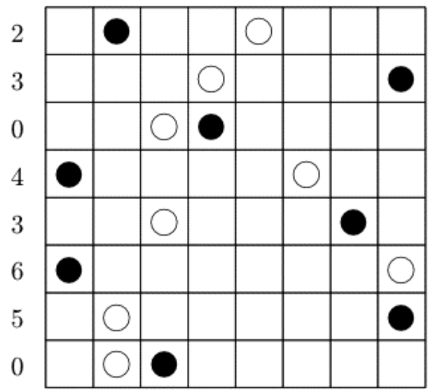

Figure 2. A Position in Northcott's Game.

This can seem an aimless game if you don't see the point, and indeed it usually goes on forever if it is played badly. But when you realize that it's only Nim in disguise once more, you'll soon be able to beat anybody pretty quickly. To the left of the board in Fig. 2 we have shown the numbers of spaces between the two pieces in each row. When someone moves, just one of these numbers will be changed, and might be either increased or decreased, according as the move was retreating or advancing. But just as in Poker-Nim. any moves increasing one of the numbers can be reversed by the next player, and so are not much use.

Who wins in Fig. 2? We can see the zero-position 2, 4, 6 among the numbers shown, and of course the two numbers 3 form another zero. Neglecting the two rows that are already 0, the only other number is 5, and we maintain that the first player can win by moving so as to reduce this 5 to 0. Whenever the other player enlarges some gap by retreating, the first player should reduce it again by the same extent. In fact the winner should *always advance* on his opponent, *never retreat*.

It should not be thought that the moves we advise here are the only good ones. For example, from Fig. 2 instead of reducing 5 to 0, we could replace 6 by 3, 4 by 1 or even 3 by 6 in the second row (for White) or 0 by 5 in the last row (for Black). In fact it will help to avoid revealing the strategy if you do *not* always reply to a retreating move by the corresponding advance—for similar reasons occasional retreating moves might be desirable.

Bogus Nim-Heaps and the Mex Rule

Consider the impartial game

$$G = \{*0, *1, *2, *5, *6, *9 \mid *0, *1, *2, *5, *6, *9\}.$$

This is a new kind of Nim-heap from which either player can move to a heap of size 0, 1, 2, 5, 6 or 9. In other words, we can regard it as a rather peculiar Nim-heap of size 3 (the first missing number) from which, as well as the usual moves to heaps of sizes 0 or 1 or 2, we are allowed to move to a heap of size 5 or 6 or 9. However, the Poker-Nim Argument shows that this extra freedom is in fact of no use whatever. To be more precise, suppose some player has a winning strategy in the game $*3 + H + K + \ldots$. Then in the same circumstances he has one in $G + H + K + \ldots$. When his strategy calls for a move in any of $*3$, H, K, …, that move is still available, and he need not use the new permitted moves from G to $*5$, $*6$ or $*9$. If his opponent tries to do so, he can immediately reverse the effect of this move by moving back to $*3$ (since 5, 6 and 9 are all greater than 3), and revert to the original strategy. So G can be replaced by $*3$ without affecting either player's chances.

The same argument shows that any game of the form

$$G = \{*a, *b, *c, \ldots \mid *a, *b, *c, \ldots\},$$

in which the same numbers appear on both sides, is really a Nim-heap in disguise. For if m is the least number from 0, 1, 2, 3, … that does *not* appear among the numbers a, b, c, …, then either player can still make from G any of the moves to $*0$, $*1$, $*2$, …, $*(m-1)$ that he could make from $*m$. If his opponent makes any other move from G, it must be to some $*n$ for which $n > m$, and can be reversed by moving back from $*n$ to $*m$. So G is really just a bogus Nim-heap $*m$.

We summarize:

> If Left and Right have exactly the same options from G,
> all of which are Nim-heaps $*a$, $*b$, $*c$, …,
> then G can itself be regarded as a Nim-heap, $*m$,
> where m is the least number 0 or 1 or 2 or …
> that is *not* among the numbers $*a$, $*b$, $*c$, … .

THE MINIMAL-EXCLUDED (MEX) RULE

This **mi**nimal-**ex**cluded number is called the **mex** of the numbers a, b, c, … .

The Sprague-Grundy Theory for Impartial Games

The above result enables us to show that *every* impartial game can be regarded as a bogus Nim-heap. For suppose we have an impartial game

$$G = \{A, B, C, \ldots \mid A, B, C, \ldots\}.$$

Then A, B, C, … are simpler impartial games, and therefore we can suppose they have already been shown to be equivalent to Nim-heaps $*a$, $*b$, $*c$, … . But in this case G can be thought of as the Nim-heap $*m$ defined above. This gives us

THE BOGUS NIM-HEAP PRINCIPLE

> Every impartial game is just a bogus Nim-heap
> (that is, a Nim-heap with reversible
> moves added from some positions).
> The Mex Rule gives the size of the heap for G as
> the least possible number that is not the size of
> any of the heaps corresponding to the options of G.

This principle was discovered independently by R. P. Sprague in 1936 and P. M. Grundy in 1939, although they did not state it in quite this way. This means that provided we can play the game of Nim, we can play *any* other impartial game given only a “dictionary” saying which **nimbers** (i.e. Nim-heaps) correspond to the positions of that game. Here’s a game played with a White Knight that gives a simple example of this dictionary method.

The White Knight

The White Knight has, from any position on the chessboard, the moves shown in Fig. 3. You may recall that he was in the habit of losing his belongings. Alice has kindly boxed them up and the boxes now form the Nim-heap to the right of the figure. Now consider the game in which you can *either* move the Knight to one of the four places shown *or* steal some of the boxes. The game ends only when the Knight is on one of the four home squares and all the boxes have gone.

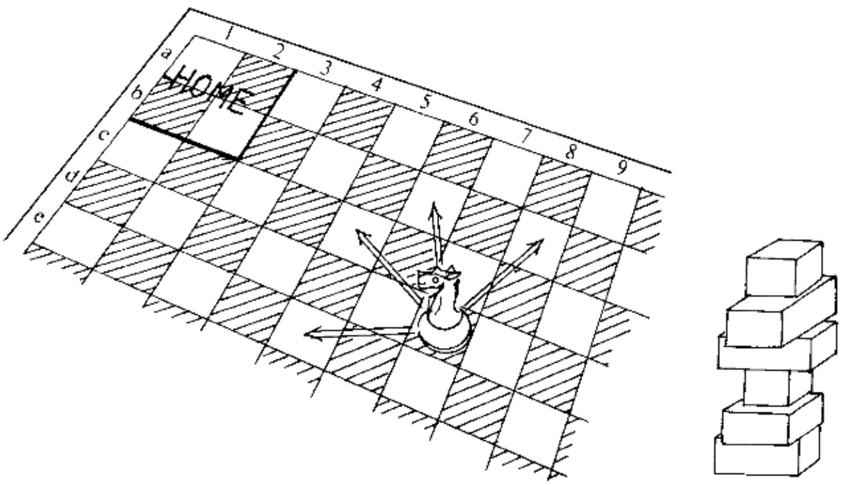

Figure 3. The White Knight and his Baggage.

The whole game is the result of adding a Nim-heap $*6$ to a game with only the Knight. Table 1 shows which nimbers correspond to the game with the Knight in various positions. Let's find the value of the Knight on d7 as in Fig. 3, assuming we already know the values of the four places he can move to. Figure 4 shows that these places can be thought of as bogus Nim-heaps of sizes

$$0, \quad 3, \quad 0, \quad 1 \quad (\text{mex} = 2)$$

and so the present position corresponds to a bogus Nim-heap of size 2, value $*2$. So the good move in Fig. 3 is to steal all but two of the boxes.

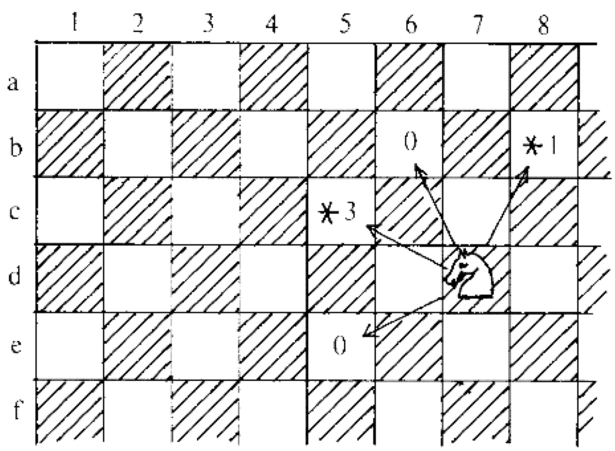

Figure 4. What the White Knight Moves are Worth.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a	0	0	*1	*1	0	0	*1	*1	0	0	*1	*1	0	0	*1	*1	0	0	*1	*1
b	0	0	*2	*1	0	0	*1	*1	0	0	*1	*1	0	0	*1	*1	0	0	*1	*1
c	*1	*2	*2	*2	*3	*2	*2	*2	*3	*2	*2	*2	*3	*2	*2	*2	*3	*2	*2	
d	*1	*1	*2	*1	*4	*3	*2	*3	*3	*3	*2	*3	*3	*3	*2	*3	*3	*3	*2	
e	0	0	*3	*4	0	0	*1	*1	0	0	*1	*1	0	0	*1	*1	0	0		
f	0	0	*2	*3	0	0	*2	*1	0	0	*1	*1	0	0	*1	*1	0	0		
g	*1	*1	*2	*2	*1	*2	*2	*2	*3	*2	*2	*2	*3	*2	*2	*2	*3			
h	*1	*1	*2	*3	*1	*1	*2	*1	*4	*3	*2	*3	*3	*3	*2	*3	*3			
i	0	0	*3	*3	0	0	*3	*4	0	0	*1	*1	0	0	*1	*1				
j	0	0	*2	*3	0	0	*2	*3	0	0	*2	*1	0	0	*1	*1				
k	*1	*1	*2	*2	*1	*1	*2	*2	*1	*2	*2	*2	*3	*2						
l	*1	*1	*2	*3	*1	*1	*2	*3	*1	*1	*2	*1								
m	0	0	*3	*3	0	0	*3	*3	0	0										
n	0	0	*2	*3	0	0	*2	*3												
o	*1	*1	*2	*2	*1	*1														
p	*1	*1	*2	*3																
q	0	0																		

Table 1. Nimbers for the White Knight.

Adding Nimbers

We saw in Chapter 2 that a Nim-heap of size 2 together with one of size 3 is equivalent to one of size 1. We now see that this was no accident, for the sum of *any* two Nim-heaps $*a$ and $*b$ is an impartial game, and so equivalent to *some* other Nim-heap $*c$. The number c is called the **nim-sum** of a and b, and written $a \overset{*}{+} b$. How can we work out nim-sums in general?

The options from $*a + *b$ are all the positions of the form $*a' + *b$ or $*a + *b'$ in which a' denotes any number (from 0, 1, 2, ...) less than a, and b' any number (from 0, 1, 2, ... again) less than b. So $a \overset{*}{+} b$ is the least number 0, 1, 2, ... not of either of the forms

$$a' \overset{*}{+} b, \quad a \overset{*}{+} b' \quad (a' < a,\ b' < b)$$

Table 2 was computed using this rule. For example the entry $6 \overset{*}{+} 3$ was computed as follows. The earlier entries 3, 2, 1, 0, 7, 6 in column 3 correspond to the options $*6' + *3$ (where $6'$ means one of 0, 1, 2, 3, 4, 5) and the earlier entries 6, 7, 4 in row 6 correspond to options

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

Table 2. A Nim-Addition Table.

$*6 + *3'$ ($3'$ means 0, 1 or 2). The least number not observed earlier in either row or column is 5, so $6 \overset{*}{+} 3 = 5$, i.e. $*6 + *3 = *5$. It might help you to follow how the table is computed if you look at the game in which our White Knight is replaced by a White Rook which can only move North or West.

You'll find a general Nim-Addition Rule in the Extras, and will have many opportunities to apply it; for example, in Chapters 4, 12, 14 and 15.

Wyt Queens

In the game of Wyt Queens any number of Queens can be on the same square and each player, when it is her turn to move, can move any single Queen an arbitrary distance North, West or North-West as indicated, even jumping over other Queens.

Because the Queens move independently, we can regard the whole game as the sum of smaller ones with just one Queen. The various Queens on the board will therefore correspond to nim-heaps $*a$, $*b$, $*c$, ... which we can add using the Nim-Addition Rule. Try computing the nimber dictionary for this game—when you get tired you can look in the Extras for more information.

The one-Queen game is a transformation of **Wythoff's Game** (1905) played with two heaps in which the move is to reduce *either* heap by *any* amount, or *both* heaps by the *same* amount. We'll meet Wyt Queens again in Chapters 12 and 13.