Nim

Nim: finitely many piles of coins; a move comprises removing a positive number of coins from a single pile; a player loses if they can't move.

Remark:
For any nim position P, either it can be won by the player with the move, or it can be won by the player without the move.

i.e. one of the two players has a "winning strategy", a way to play which guarantees a win.

The "nim sum", $n \oplus m$, of natural numbers n and m is the result of writing the binary expansions of n and m and "adding without carrying". (In computer science, this is called "XORing the bitstrings"; in many programming languages, it's written as "`n^m`".)

Theorem:
The player without the move can win from the Nim position with piles of sizes $n_1, ..., n_k$ iff $n_1 \oplus n_2 \oplus ... \oplus n_k = 0$

Proof:
Suppose inductively that this is true for all nim positions with fewer coins involved.

First, suppose
$$n_1 \oplus n_2 \oplus ... \oplus n_k = b \neq 0.$$
We show that we can win if we have the move.

Consider binary expansions.
Some n_i has a 1 in the same position as the leading 1 of b,
so
$$n_i \oplus b < n_i.$$

So we can move by taking coins from the ith pile so as to leave $n_i(+)b$ coins in that pile.

Then in the new position, the nim sum of the pile sizes is
$$n_1 \oplus ... \oplus n_{i-1} \oplus n_i \oplus b \oplus n_{i+1} \oplus ... \oplus n_k$$
$$= b \oplus b$$
$$= 0$$

So by the induction hypothesis, the player without the move wins from here. But that's us!

Now suppose
$$n_1 \oplus n_2 \oplus ... \oplus n_k = 0$$
and we don't have the move.

If our opponent can't move, we've won.
Else, suppose they move by taking coins from the ith pile, leaving $m < n_i$.
But then $m \oplus n_i \neq 0$, so

$$n_1 \oplus ... \oplus m \oplus ... \oplus n_k \neq n_1 \oplus ... \oplus n_i \oplus ...n_k = 0,$$

so by the induction hypothesis, we're left with a position won by the player with the move, which is us.