Multisets

Example:
A bag of Scrabble tiles contains 100 tiles: 10 A's, 2 B's, 2 C's, 5 D's and so on.

When you start a game, you take 7 letters from the bag, and put them on a rack. How many possible hands can you get, if we say that the order of the tiles on the rack matters? How about if it doesn't?

A multiset is a "set with multiplicity".

Notation: $\{2 * a, 3 * b, 1 * c\}$

(Think of a "bag" with 2 a's, 3 b's and a c in it.)

We also allow "infinite multiplicity", denoted $\{\infty * a\}$.

Multiplicities are also called "repetition numbers".

The size of a multiset is the sum of the multiplicities (may be ∞).

An r-permutation of a multiset is an ordered list of r elements from the multiset;
e.g. the 2-permutations of $\{2 * a, 1 * b\}$ are
aa, ab, ba;
the 3-permutations of $\{\infty * a, 2 * b\}$ are
$aaa, aab, aba, baa, abb, bab, bba$.

A permutation of a multiset of size n is an n-permutation.

Example: how many permutations are there of the unfortunate scrabble hand $\{4 * U, 1 * J, 2 * K\}$?

Theorem:
Let S be a multiset with k types with finite multiplicities $n_1, ..., n_k$.
Let $n = \Sigma_i n_i$ be the size of S.
Then the number of permutations of S is
$n!/n_1! * n_2! * ... * n_k!$.

Proof:
Label the elements $1, ..., n$. Each permutation of $\{1, ..., n\}$ yields a permutation of S, and two yield the same permutation precisely when we can get one from another by permuting the labels on elements of the same type.
So there are $n_1! * n_2! * ... * n_k!$ permutations of $\{1, ..., n\}$ per permutation of S. We conclude by the division principle.

Example:
We have 4 black rooks and 4 white rooks. How many ways are there of putting them on a chess board such that no two are attacking (/defending) each other? e.g.

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.......R  8
....r...  5
..R.....  3
......R.  7
...r....  4
.R......  2
.....r..  6
r.......  1
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Solution:
First, just choose the 8 squares for them to occupy.

By listing off the filled columns row-by-row, a choice corresponds to a permutation of the columns, so there are 8!.

For a given such choice, a choice of colours corresponds to a permutation of the multiset $\{4 * r, 4 * R\}$.

So the answer is

$$8! * (8!/4! * 4!) = 2822400$$

An <u>r-submultiset</u>, or <u>r-combination</u>, of a multiset S is a multiset S' of size r such that for all x,
the multiplicity of x in S' is at most the multiplicity of x in S.

e.g. the 2-submultisets of $\{3 * a, b\}$ are $\{2 * a\}$, $\{1 * a, 1 * b\}$.

Theorem:
The number of r-combinations of a multiset with k types each with multiplicity at least r is

$$\binom{r+k-1}{r}$$

Example:
If we have a bag containing red, green and blue marbles, with many of each, and we draw 5 marbles from the bag, how many possible results (numbers of each colour drawn) are there?
Answer: $\binom{5+3-1}{5} = \binom{7}{5} = 7!/2!5! = 21$

Proof:
We can identify an r-submultiset with an arrangement of $k-1$ partitions interspersed among r identical objects, by counting the numbers of objects between the partitions; e.g. with $k = 6$ and $r = 8$

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oo|ooo||o|oo|
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corresponds to

$$\{2 * a_1, 3 * a_2, 0 * a_3, 1 * a_4, 2 * a_5, 0 * a_6\}.$$

These arrangements correspond to choosing r of the $r + k - 1$ characters to be 'o's, so the number of such arrangements is $\binom{r+k-1}{r}$.