Pigeonhole Principles

Pigeonhole Principle (PP):
If some pigeons are in some pigeonholes,
and there are fewer pigeonholes than there are pigeons,
then some pigeonhole must contain at least two pigeons.

// The "pigeons" and "pigeonholes" can be abstract!

Example:
If there are 367 people in a room,
there must be two who share a common birthday.

Interlude: maps and numbers

$f : X \to Y$ map between finite sets.

For $y \in Y$, $f^{-1}(y) =$ "fibre of f over y" $= \{x \mid f(x) = y\}$.

Recall:

- f is surjective aka onto,
  written $f : X \twoheadrightarrow Y$,
  if for all $y \in Y$, $|f^{-1}(y)| \geq 1$
- f is injective aka 1-1,
  written $f : X \hookrightarrow Y$,
  if for all $y \in Y$, $|f^{-1}(y)| \leq 1$
- f is bijective aka a (1-1) correspondence aka invertible,
  written $f : X \xrightarrow{\equiv} Y$,
  if f is both injective and surjective,
  i.e. if for all $y \in Y$, $|f^{-1}(y)| = 1$

Remark:
If f is

- injective then $|X| \leq |Y|$ (Pigeonhole principle)
- surjective then $|X| \geq |Y|$
- bijective then $|X| = |Y|$

Applications of the Pigeonhole principle

Example:
If I take 13 coins,
divide them into 9 piles, placed in a row,
then there will be a group of neighbouring piles within the row such that

there are exactly 4 coins in the group.
(Generally: n coins, m piles; must be k coins in a contiguous group if

$n + k < 2 * m$

(this isn't sharp))

Proof:
Let $a_i :=$ number of coins in first i piles, $1 \leq i \leq 9$.
Consider the 18 numbers

$a_1, a_2, ..., a_9, a_1 + 4, a_2 + 4, ..., a_9 + 4.$

Since $1 \leq a_i \leq 13$, these numbers are all between 1 and 17.
So by the PP, two must be equal.

Since no two a_i are equal (since the piles are non-empty),
and similarly no two $a_i + 4$ are equal,
we must have $a_i = a_j + 4$ for some i,j.

So $a_i - a_j = 4$,
so 4 is the sum of the sizes of the piles after i and up to j,
namely piles $i + 1, ..., j$.

Example:
Using as many coins as I want,
I make a row of k piles.
Then there is a group of neighbouring piles such that the number of coins in the group is divisible by k.

Proof:
Let $a_1, ..., a_k$ be as above.

Let r_i be the remainder on dividing a_i by k.
If any $r_i = 0$, we're done.
Else, $0 < r_i < k$,
so by the PP, two remainders are equal,
$r_i = r_j$.
But then $a_i - a_j$ is divisible by 7, and we conclude as in the previous example.

Packed Pigeonhole Principle

Packed Pigeonhole Principle (PPP):
If there are more than $k * n$ pigeons in n pigeonholes,
then some pigeonhole contains more than k pigeons.
(Note: "Packed" is not standard terminology. This principle is commonly referred to as the pigeonhole principle. Brualdi calls something slightly more general (but less pleasing) the "strong pigeonhole principle", but I don't think we need to cover it)

Example:
If $a_1, ..., a_{n^2+1}$ is a sequence of $n^2 + 1$ real numbers,
there is a subsequence of length $n + 1$ which is monotonic, i.e. is either (nonstrictly) increasing or (nonstrictly) decreasing.

Proof:
Suppose there is no increasing subsequence of length $n + 1$.

Let l_i be the length of the longest increasing subsequence starting with a_i.
So $1 \leq l_i \leq n$.

So by the PPP, $n + 1$ of these $n^2 + 1$ numbers are equal;
say $l_{i_1} = ... = l_{i_{n+1}}$.

Now suppose $a_{i_j} < a_{i_{j+1}}$.
Then we can extend the longest increasing subsequence starting with $a_{i_{j+1}}$ to a longer one starting with a_{i_j}, by prepending a_{i_j}.
This contradicts $l_{i_j} = l_{i_{j+1}}$.

So $(a_{i_j})_j$ is a decreasing sequence of length $n + 1$.

Abstract version:
If $f : X \twoheadrightarrow Y$ is a surjection,
and if all fibres are of size at most k,
i.e. $|f^{-1}(y)| \leq k$ for all y,
then $|X| \leq k|Y|$.

Remark (Division principle, map form):
If $f : X \twoheadrightarrow Y$ is a surjection,
and if all fibres have size exactly k,
i.e. $|f^{-1}(y)| = k$ for all y,
then $|X| = k|Y|$.

(Proof: partition X according to the value of f, apply division principle)

Averaging principle:
Given integers $a_1, ..., a_n$,
some a_i is at least the average,
$a_i \geq (a_1 + ... + a_n)/n$
(Note the average might not be an integer!)

Packed Pigeonhole follows from averaging:
if there are more than $k * n$ pigeons,
then the average number of pigeons per pigeonhole is more than k;
some pigeonhole has at least the average number of pigeons,
so has more than k pigeons.

Example:
Discs (p.75)