Binomial coefficients

Miscellaneous Curiosities

Recall:

- For n a non-negative integer and r an integer,
  $\binom{n}{r}$ = number of subsets of size r of a set of size n
  $= \frac{n!}{n!(n-r)!}$ if $0 \le r \le n$
  $= 0$ else.
- Pascal's triangle
- Pascal's Formula: $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$
- $\binom{n}{r} = \binom{n}{n-r}$
- $\sum_{r=0}^{n} \binom{n}{r} = 2^n$
- $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$

Remark:
$\binom{n}{r}$ = number of paths from root of Pascal's triangle to the (n, r) position.

Further identities:

- $k\binom{n}{k} = n\binom{n-1}{k-1}$ (immediate from $\binom{n}{k} = \frac{n(n-1)...(n-k+1)}{k(k-1)...1}$)
- $(x+1)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$
- $0 = ((-1)+1)^n = \sum_{r=0}^{n} \binom{n}{r}(-1)^r$;
  so alternating sum of binomial coefficients is 0;
  so sum of even coefficients = sum of odd coefficients = 2^{n-1}.

Yet further identities:

(i) $\binom{n+1}{r+1} = \sum_{s=0}^{n} \binom{s}{r}$

(ii) $\sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n}$

(iii) $\sum_{r=0}^{n} r\binom{n}{r} = n2^{n-1}$

Proofs:

(i) Iteratively apply Pascal's formula:

$$\begin{aligned}\binom{n+1}{r+1} &= \binom{n}{r+1} + \binom{n}{r} \\ &= \binom{n-1}{r+1} + \binom{n-1}{r} \\ &= ... \\ &= \binom{0}{r+1} + \binom{0}{r} + ... + \binom{n-1}{r} \\ &= \binom{0}{r} + ... + \binom{n-1}{r}\end{aligned}$$

Alternative inductive proof:
Easily holds for $n = r = 0$.
Suppose inductively it holds for smaller $n + r$.
Then using Pascal's formula, we have:
$\binom{n+1}{r+1} = \binom{n}{r+1} + \binom{n}{r}$
$= \sum_{k=0}^{n-1} \binom{k}{r} + \sum_{k=0}^{n-1} \binom{k}{r-1}$
$= \sum_{k=0}^{n-1} (\binom{k}{r} + \binom{k}{r-1})$
$= \sum_{k=0}^{n-1} \binom{k+1}{r}$
$= \sum_{k=1}^{n} \binom{k}{r}$
$= \sum_{k=0}^{n} \binom{k}{r}$

(ii) Consider diamonds in Pascal's triangle.

OR: Given a set S of size $2n$, arbitrarily split it into two sets S_1, S_2 of size n.
Then an n-subset S' of S corresponds to the pair $(S' \cap S_1, S' \cap S_2)$.
The pairs of subsets arising in this way are precisely those of sizes summing to n,
so

$$\binom{2n}{n} = \sum_{r=0}^{n} \binom{n}{r}\binom{n}{n-r} = \sum_{r=0}^{n} \binom{n}{r}^2$$

(iii) Neat algebraic proof:
$n(x+1)^{n-1} = \frac{d}{dx}(x+1)^n = \frac{d}{dx} \sum_{r=0}^{n} \binom{n}{r} x^r$
$= \sum_{r=0}^{n} r \binom{n}{r} x^r$.
This holds for all x; taking $x = 1$ gives the result.

Examples of (i):

- $\binom{n}{1} = \sum_{s=0}^{n-1} \binom{s}{0} = \sum_{s=0}^{n-1} 1 = n$
- $\binom{n}{2} = \sum_{s=0}^{n-1} \binom{s}{1} = \sum_{s=0}^{n-1} s = n$th triangular number
- $\binom{n}{3} = \sum_{s=0}^{n-1} \binom{s}{2} = n$th pyrimidal number

Multinomial theorem

What is the coefficient $a_{r,s,t}$ of $x^r y^s z^t$ in the expansion of $(x + y + z)^n$?

Clearly $a_{r,s,t} \neq 0$ only if $r + s + t = n$.

$a_{r,s,t}$ is the number of ways of choosing r x's, s y's, and t z's from the n factors $(x + y + z)$;
i.e. the number of strings like "xyzzyxyzzy" with this many of each letter;
i.e. the number of permutations of the multiset $\{r * x, s * y, t * z\}$.

So as we saw before,
$a_{r,s,t} = \frac{n!}{r!s!t!}$.

Write $\binom{n}{r\ s\ t}$ for this number.

Generalising to arbitrarily many variables, we have

Theorem:
$(x_1 + ... + x_t)^n = \sum_{n_i \geq 0, n_1+...+n_t=n} \binom{n}{n_1\ n_2\ ...\ n_t} x_1^{n_1} x_2^{n_2} ... x_t^{n_t}$

Here, $\binom{n}{n_1\ n_2\ ...\ n_t} = \frac{n!}{n_1!...n_t!}$ are the multinomial coefficients (only defined if $n_1 + ... + n_t = n$).

Note:
$\binom{n}{r} = \binom{n}{r\ n-r}$.

The number of terms in the multinomial expansion of $(x_1 + ... + x_t)^n$ is the number of n-combinations with t types in unlimited supply, which we saw is

$$\binom{n+t-1}{n}.$$

Unnatural exponents: $(x + y)^\alpha$

Theorem [Newton's Binomial Theorem]:
Let α be a real (or even complex) number.
Suppose $0 \leq |x| < |y|$.
Then

$$(x + y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-k}$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)...(\alpha-k+1)}{k!}.$$

Note that for α natural, this agrees with our previous definition.

Proof (not on syllabus):
Dividing through by y^α, sufficient to show that for $|z| < 1$,

$$(1 + z)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k.$$

We show this for complex z with $|z| < 1$.
$(1 + z)^\alpha = \exp(\alpha \log(1 + z))$ for any choice of branch.
This is holomorphic on the domain $|z| < 1$,
so the Taylor series at 0 converges to the value of the function on this domain.

Since $\frac{d}{dz}(1 + z)^\alpha = \alpha(1 + z)^{\alpha-1}$ and $(1 + 0)^\alpha = 1$,
this gives

$$\begin{aligned}(1 + z)^\alpha &= ((1 + z)^\alpha|_{z=0})\tfrac{z^0}{0!} + \\ &\quad (\alpha(1 + z)^{\alpha-1}|_{z=0})\tfrac{z^1}{1!} + \\ &\quad (\alpha(\alpha-1)(1 + z)^{\alpha-2}|_{z=0})\tfrac{z^2}{2!} + ... \\ &= \sum_{k=0}^{\infty}(\alpha(\alpha-1)...(\alpha-k+1))\tfrac{z^k}{k!} \\ &= \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k\end{aligned}$$

□

Examples:

- $\frac{1}{1+z} = (1 + z)^{-1}$
  $= \sum_{k=0}^{\infty} \binom{-1}{k} z^k$

$= \sum_{k=0}^{\infty} \frac{(-1)*(-2)*...*(-k)}{k*(k-1)*...*1} z^k$
$= \sum_{k=0}^{\infty} (-1)^k z^k$
$= 1 - z + z^2 - z^3 + ...$

- $$\sqrt{37} = \sqrt{6^2+1} = 6\sqrt{1+1/36} = 6(1+1/36)^{1/2}$$
  $$= 6(\textstyle\sum_{k=0}^{\infty} \binom{1/2}{k} (1/36)^k)$$

  Now for $k > 0$,

  $$\binom{1/2}{k} = \frac{\frac{1}{2}\frac{1-2}{2}\cdots\frac{1-2(k-1)}{2}}{k!}$$
  $$= \frac{(-1)^{k-1} 1*3*5*...*(2k-3)}{2^k k!}$$
  $$= \frac{(-1)^{k-1}(2k-2)!}{2^k (2*4*...*(2k-2))k!}$$
  $$= \frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}(k-1)!k!}$$
  $$= \frac{(-1)^{k-1}}{k2^{2k-1}}\binom{2k-2}{k-1}$$

  So

  $$\sqrt{1+z} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k2^{2k-1}}\binom{2k-2}{k-1} z^k$$
  $$= 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \frac{1}{25}z^4 + ...$$

  So $\sqrt{37} = 6(1+1/36)^{1/2}$
  $\approx 6(1 + 1/(2*36) - 1/(8*36^2) + 1/(16*36^3) - 1/(25*36^4))$
  $= 6.0828$

  (Error is very small: $(6*(1+1/(2*36) - 1/(8*36^2) + 1/(16*36^3) - 1/(25*36^4)))^2 = 36.99999992692224$)