Partial Orders

Basics

A partial order on a set X is a binary relation $\leq$ which is

- reflexive ($x = y => x \leq y$)
- transitive (if $x \leq y$ and $y \leq z$ then $x \leq z$)
- antisymmetric (if $x \leq y$ and $y \leq x$ then $x = y$).

A set X equipped with a partial order $\leq$, denoted $(X; \leq)$, is called a partially ordered set or a poset.

"$x \geq y$" means "$y \leq x$".
"$x < y$" means "$x \leq y$ and $x \neq y$".

Examples:

(i) The usual order $\leq$ on the integers.

(ii) The relation of divisibility is a partial order on the natural numbers; $(\mathbb{N}; |)$ is the corresponding poset.

(iii) If A is a set, the set of subsets of A is partially ordered by inclusion, $\subseteq$.

Hasse diagrams:
x covers y if $x > y$ and there is no z such that $x > z > y$.

The Hasse diagram of a finite poset $(X, \leq)$ consists of points for the elements of X and a line drawn upwards from y to x whenever x covers y.

(We will see below that every finite poset has a Hasse diagram.)

x is minimal if $x > y$ for no y.
x is maximal if $x < y$ for no y.

Lemma:
$<$ is transitive: if $x < y$ and $y < z$ then $x < z$.

Proof:
$x \leq z$ by transitivity of $\leq$.
Suppose $x = z$.
Then $y \leq x$ and $x \leq y$, so $x = y$ by antisymmetry, contradicting $x < y$.
So $x \neq z$. □

Lemma:
Any finite poset has at least one minimal element, and at least one maximal element.

Proof:
Suppose $(X; \leq)$ has no minimal element.
Then there exist arbitrarily long chains $x_1 > x_2 > x_3 > ... > x_n$.
By transitivity of $>$, the x_i are distinct, so we contradict finiteness. □

A partial order $\leq$ on a set X is total (aka linear) if for all x and y in X, either $x \leq y$ or $y \leq x$.

Lemma:
Any finite total order $(X; \leq)$ can be enumerated as $X = \{x_1, ..., x_n\}$ with $x_i \leq x_j$ iff $i \leq j$.

(i.e. $(X; \leq)$ is isomorphic to $\{1, ..., n\}$ with the usual order.)

Proof:
If x is minimal in a total order, then $x \leq y$ for any y.

Let x_1 be minimal in X, then let x_2 be minimal in $X \setminus \{x_1\}$,
and so on.

Then $x_i \leq x_j$ for $i \leq j$.
By antisymmetry, $x_i \not\leq x_j$ for $i \not\leq j$. □

Lemma:
Any finite poset $(X; \leq)$ can be linearised,
i.e. there exists a total order $\leq'$ such that $x \leq y => x \leq' y$.

Proof:
Let $x_1, ..., x_n$ be the minimal elements of $(X; \leq)$.
Let $X' := X \setminus \{x_1, ..., x_n\}$.
By induction, $(X'; \leq)$ can be linearised, say to $\leq'$.
Extend $\leq'$ to X by defining

- $x_i \leq' x_j$ iff $i \leq j$
- $x_i \leq' y$ for any $y \in X'$

This is total. □

Consequence:
Any finite poset has a Hasse diagram:
draw the points with heights ordered according to a linearisation of the partial order,
then draw a line whenever x covers y, which implies that x is above y.
(Nudge the points horizontally if there are any overlapping lines).

Chains and antichains

Definition:
Let $(X; \leq)$ be a poset.

$\emptyset \neq C \subseteq X$ is a chain if $(C; \leq)$ is a total order.

$\emptyset \neq A \subseteq X$ is an antichain if $a_1 \leq a_2 \Rightarrow a_1 = a_2$ for $a_i \in A$.

A chain partition is a partition $X = C_1 \cup ... \cup C_n$ by disjoint chains.

An antichain partition is a partition $X = A_1 \cup ... \cup A_n$ by disjoint antichains.

Theorem:
In a finite poset $(X; \leq)$,

(i) The maximal size of a chain is equal to the minimal size of an antichain partition.

(ii) [Dilworth's theorem] The maximal size of an antichain is equal to the minimal size of a chain partition.

Proof:
First observe that a chain and an antichain can have no more than 1 point in common,

$$|C \cap A| \leq 1.$$

So given an antichain partition and a chain,
each element of the chain is in precisely one of the antichains,
and no two elements of the chain are in the same antichain,
so

size of any chain $\leq$ size of any antichain partition

so

maximal size of a chain $\leq$ minimal size of an antichain partition.

Similarly for (ii): given a chain partition, each element of an antichain is in exactly one of the chains in the partition,
and no two elements of the antichain are in the same chain, so

maximal size of an antichain $\leq$ minimal size of a chain partition.

So it remains to see

(i) There exists an antichain partition of size the maximal size of a chain;

(ii) There exists a chain partition of size the maximal size of an antichain.

These require separate arguments.

(i) Let C be a chain of maximal length.
Say $C = \{c_1, ..., c_n\}$ with $c_i \leq c_j$ iff $i \leq j$.

For $i = 1, ..., n$, recursively define

$$A_i := \text{the set of minimal elements of } X \setminus (A_1 \cup ... \cup A_{i-1}).$$

Then A_i is an antichain,
and $c_i \in A_i$ so no A_i is empty.

If $x \in X \setminus \cup_i A_i$ then $x \geq c_i$ for all i,
so we could extend C to a larger chain by adjoining x,
contradicting maximality of C.
So $X = \cup_i A_i$.

So the A_i form an antichain partition of size $n = |C|$ as required.

(ii) By induction on the size of X.

First, suppose some antichain A of maximal size m is not the set of maximal elements and is not the set of minimal elements.

Let

$$A^+ := \bigcup_{a \in A} \{x \mid x \geq a\}$$
$$A^- := \bigcup_{a \in A} \{x \mid x \leq a\}.$$

Then $A^+ \cup A^- = X$ by maximality of A,
and $A^+ \cap A^- = A$.

Now A is a maximal-size antichain in A^+,
and $A^+ \neq X$ since A is not the set of maximal elements,
so by induction, A^+ has a chain partition of size $|A|$.

Similarly, we have a chain partition of A^-.

For each element $a \in A$, a is in one of the chains of A^+ and one of the chains of A^-, and the union of these two chains is a chain C_a in X.

Then $\{C_a \mid a \in A\}$ is a chain partition of X of size $m = |A|$.

For the remaining case,
suppose every maximal-size antichain is either the set of maximal elements or the set of minimal elements.

Let x be minimal and y be maximal, with $x \leq y$.

(To see that such that such x and y exist:
we proved above that maximal and minimal elements always exist,
so we only need to see that some minimal element is comparable with (and hence $\leq$) some maximal element.
Otherwise, the minimal elements and the maximal elements together form an antichain,
which contradicts our assumption unless the set of minimal elements is equal to the set of maximal elements,
in which case we can take $x = y$.)

Then $X \setminus \{x, y\}$ has no antichains of size m but has an antichain of size $m - 1$,
so by induction it has a chain partition of size $m - 1$.
Adjoining the chain $\{x, y\}$, we obtain a chain partition of X of size m.

□

Bonus: Sperner's Theorem

Example:
Inductively define "symmetric" chain partitions S_n of the set of subsets of $\{1, ..., n\}$:

Let S_1 be the partition with only one chain,
$\emptyset \subsetneq \{1\}$.

Given a chain partition S_n of $\{1, ..., n\}$,
let S_{n+1} have, for each chain $A_1 \subsetneq ... \subsetneq A_k$ of S_n,

- the chain $A_1 \subsetneq ... \subsetneq A_k \subsetneq A_k \cup \{n+1\}$,
- and, if $k > 1$, the chain $A_1 \cup \{n+1\} \subsetneq ... \subsetneq A_{k-1} \cup \{n+1\}$.

Each chain $A_1 \subsetneq ... \subsetneq A_k$ in S_n has subsequent elements differing in size by one,
and $|A_1| + |A_k| = n$.

Hence each chain contains a subset of size $\lfloor \frac{n}{2} \rfloor$,
so $|S_n| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

($\lfloor \frac{n}{2} \rfloor = n/2$ "rounded down")

So by Dilworth, we obtain "Sperner's Theorem":
The set of subsets of $\{1, ..., n\}$ of size $\lfloor \frac{n}{2} \rfloor$ is a maximal-size antichain.