Number sequences

A number sequence is simply an infinite sequence $h_0, h_1, h_2, ...$ of numbers. For us, h_i will typically be an integer.

Examples:
1,2,3,4,5,...
2,4,8,16,32,...
2,3,5,7,13,...
1,1,2,3,5,8,13,...
1,5,10,10,5,1,0,0,0,0,...

Generating functions

The generating function of a number sequence $h_0, h_1, ...$ is the formal power series $g(x) = \sum_{n=0}^{\infty} h_n x^n$.

Technical remark:
Despite the notation and terminology, we do not assume any convergence;
we do not need $g(a)$ to make sense for a a real number,
so g doesn't really have to be a function in the usual sense.
for example, $\sum_{n=0}^{\infty} n^n x^n$ doesn't converge for $x \neq 0$,
but it's a perfectly good generating function.

We use the usual algebraic notation for generating functions. We can make sense of algebraic operations as follows:

Given formal power series $g(x) = \sum_{n=0}^{\infty} h_n x^n$ and $g'(x) = \sum_{n=0}^{\infty} h'_n x^n$, and a number c, we define

$$g(x) + g'(x) := \sum_{n=0}^{\infty} (h_n + h'_n) x^n$$
$$cg(x) := \sum_{n=0}^{\infty} c h_n x^n$$
$$g(x)g'(x) := \sum_{n=0}^{\infty} (\sum_{j=0}^{n} h_j h'_{n-j}) x^n.$$

We also write $c(x) = \frac{a(x)}{b(x)}$ to mean that $a(x) = b(x)c(x)$ (this is well-defined).

We can often use this algebraic structure to write generating functions compactly.

Example 1:
Consider the binomial coefficients $\binom{m}{n}$ for a fixed m.
This is a finite number sequence, but we can make it infinite by appending 0s,

$$\binom{m}{0}, \binom{m}{1}, ..., \binom{m}{m}, 0, 0, 0, ...$$

So the generating function is

$$\binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 ... + \binom{m}{m}x^m$$
$$= (x+1)^m$$

Example 2:

The generating function of the number sequence

$1, 1, 1, ...$

is $g(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + ...$

Now, multiplying out,

$(1 + x + x^2 + ...)(1 - x) = 1 + (-1 + 1)x + (-1 + 1)x^2 + ...,$

so $g(x) = 1/(1 - x)$.

□

Generating functions provide an efficient notation for describing and manipulating classes of combinatorial problems.

Example 3:

Given t, let h_n be the number of n-combinations of a multiset with t types and infinite multiplicity for each type.

We know $h_n = \binom{n+t-1}{t-1}$, so the generating function is

$g(x) = \sum_{n=0}^{\infty} h_n x^n = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n.$

But note also that

$g(x) = (1 + x + x^2 + ...)^t,$

since when we multiply the right hand side out,

the coefficient of x^n is precisely the number of ways of obtaining x^n as $x^{e_1}x^{e_2}...x^{e_t}$,

which is the number of solutions in non-negative integers to $e_1 + ... + e_k = n$, which (as we've seen before) is h_n.

So as in the previous example,

$g(x) = (1 + x + x^2 + ...)^t = \left(\frac{1}{1-x}\right)^t = \frac{1}{(1-x)^t}.$

□

Note we found here the power series expansion of $\frac{1}{(1-x)^t}$, which will come in handy later.

Lemma 1:

$\frac{1}{(1-x)^t} = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n$

If we have restrictions on how many of each type we're allowed to take in a combination, we can incorporate these into an algebraic expression for the generating function.

Example 4:

Find the generating function for the number h_n of bags of n marbles consisting of an even number of red marbles, at least 1 green marble, at most 36 blue marbles, and an odd number of yellow marbles.

Arguing as in the previous example, the generating function is

$g(x) = (1+x^2+x^4+...)(x+x^2+x^3+...)(1+x+x^2+...+x^36)(x+x^3+x^5+...)$

$$= \frac{1}{1-x^2}\frac{x}{1-x}\frac{1-x^{37}}{1-x}\frac{x}{1-x^2}.$$
$$= \frac{x^2(1-x^{37})}{(1-x^2)^2(1-x)^2}.$$

Example 5:
Find the generating function for the number h_n of bags of n marbles consisting of an even number of red marbles, a multiple of 3 of green marbles, at most 2 blue marbles, and at most one yellow marble. Hence explicitly determine h_n.

$$\begin{aligned} g(x) &= (1 + x^2 + x^4 + ...)(1 + x^3 + x^6 + ...)(1 + x + x^2)(1 + x) \\ &= \frac{1}{1-x^2}\frac{1}{1-x^3}\frac{1-x^3}{1-x}(1 + x) \\ &= \frac{1+x}{(1-x^2)(1-x)} \\ &= \frac{1+x}{(1+x)(1-x)(1-x)} \\ &= \frac{1}{(1-x)^2} \\ &= \sum_{n=0}^{\infty} \binom{n+2-1}{2-1} x^n \text{ (by Lemma 1)} \\ &= \sum_{n=0}^{\infty} (n + 1)x^n. \end{aligned}$$

So there are $n + 1$ such bags of n marbles!

(Exercise: find a direct proof of this, without going via generating functions.)

Example 6:
Find the generating function for the number h_n of ways of making n cents out of Canadian coins.

The coins in current circulation are worth 5,10,25,100, and 200 cents each.

So h_n is the number of solutions in non-negative integers to
$$5N + 10D + 25Q + 100L + 200T = n.$$
Equivalently, h_n is the number of solutions to
$$e_1 + e_2 + e_3 + e_4 + e_5 = n$$
where e_1 is a multiple of 5, e_2 is a multiple of 10, etc.

So as above,
$$\begin{aligned} g(x) &= (x^5 + x^{10} + x^{15} + ...)(x^{10} + x^{20} + ...)...(x^{200} + x^{400} + ...) \\ &= \frac{1}{(1-x^5)(1-x^{10})...(1-x^{200})}. \end{aligned}$$

Exponential Generating Functions

The exponential generating function of a number sequence $h_0, h_1, ...$ is the formal power series
$$g^{(e)}(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!}.$$

While ordinary generating functions are useful for counting combinations, exponential generating functions are useful for counting permutations.

Example 7:
The exponential generating function of

$$(m,0), P(m,1), ..., P(m,m), 0, 0, 0, ...$$

is

$$\begin{aligned} g^{(e)} &= \sum_{n=0}^{m} \frac{m!}{(m-n)!} \frac{x^n}{n!} \\ &= \sum_{n=0}^{m} \binom{m}{n} x^n \\ &= (1+x)^m \end{aligned}$$

Example 8:

Let h_n be the number of n-permutations of a multiset with k different types, each with infinite multiplicity,

$$\{\infty \cdot a_1, ..., \infty \cdot a_k\}.$$

So $h_n = k^n$.
Then the exponential generating function is : $g^{(e)}(x) = \sum_{n=0}^{\infty} \frac{k^n x^n}{n!} = e^{kx}$.

(remark for anyone who might worry what exactly we mean by this last equality: we can just define e^{ax} to be the formal power series $\sum_{n=0}^{\infty} \frac{a^n}{n!} x^n$. This obeys the usual law $e^{ax}e^{bx} = e^{(a+b)x}$. We could define more, but this will suffice for our purposes.)

Theorem:

Let h_n be the number of n-permutations of the multiset

$$S := \{n_1 \cdot a_1, ..., n_k \cdot a_k\},$$

with $n_i \in \mathbb{N} \cup \{\infty\}$.

Then the exponential generating function is

$$g^{(e)} = f_{n_1}(x) f_{n_2}(x) ... f_{n_k}(x)$$

where

$$f_n(x) = \sum_{i=0}^{n} \frac{x^i}{i!}$$

and in particular, $f_\infty(x) = e^x$.

Proof:

$$\begin{aligned} h_n &= \sum_{S' \text{ an } n\text{-combination of } S} (\text{number of permutations of } S') \\ &= \sum_{\{(m_1,...,m_k) \mid m_1+...+m_k=n,\ 0 \le m_i \le n_i\}} \\ &\quad (\text{number of permutations of } \{m_1 * a_1, ..., m_k * a_k\} \\ &= \sum_{\{(m_1,...,m_k) \mid m_1+...+m_k=n,\ 0 \le m_i \le n_i\}} \frac{n!}{m_1!...m_k!} \\ &= n! \sum_{\{(m_1,...,m_k) \mid m_1+...+m_k=n,\ 0 \le m_i \le n_i\}} \frac{1}{m_1!...m_k!} \end{aligned}$$

Meanwhile, if we multiply out $f_{n_1}(x) f_{n_2}(x) ... f_{n_k}(x)$, we find the coefficient of x^n is

$$= \sum_{\{(m_1,...,m_k) \mid m_1+...+m_k=n,\ 0 \le m_i \le n_i\}} \frac{1}{m_1!...m_k!}.$$

So this is indeed the exponential generating function.
□

Just as we saw with ordinary generating functions, if we have restrictions on how many of each type we are allowed in a permutation, we can incorporate these restrictions into the factors in the above expression for the exponential generating function, by only including the appropriate powers of x.

Often, expanding out the resulting power series will give us a solution to the combinatorial problem, as the following example demonstrates.

Example 9:

How many n-digit numbers can be written using only the digits '1','2', and '3', using an even number of '2's and at least 1 '3'?

The exponential generating function is

$$g^{(e)}(x) = (\sum_{n=0}^{\infty} \frac{x^n}{n!})(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!})(\sum_{n=1}^{\infty} \frac{x^n}{n!})$$
$$= (e^x)(\frac{e^x+e^{-x}}{2})(e^x - 1)$$
$$= \frac{1}{2}(e^{3x} + e^x - e^{2x} - 1)$$
$$= \sum_{n=1}^{\infty} \frac{3^n+1-2^n}{2n!}$$

So the answer is $\frac{3^n+1-2^n}{2}$