Solutions to Test 1

#1 Lets count the number of n with $1 \le n \le 100$ such that $m^2 | n$ for some $m > 1$. It is enough to look at primes p such that $p^2 | n$ and the only possible p's given that $n \le 100$ are

$p = 2, 3, 5, 7$

$p = 2$: There are 25 numbers ≤ 100 divisible by 4.

$p = 3$: There are 11 numbers ≤ 100 divisible by 9 and 2 of them are divisible by 4.

$p = 5$: There are 4 numbers ≤ 100 divisible by 25 and 1 of them is divisible by 4.

$p = 7$: Two numbers ≤ 100 are divisible by 49.

Altogether then $25 + 11 - 2 + 4 - 1 + 2 = 39$ numbers ≤ 100 are divisible by some square.

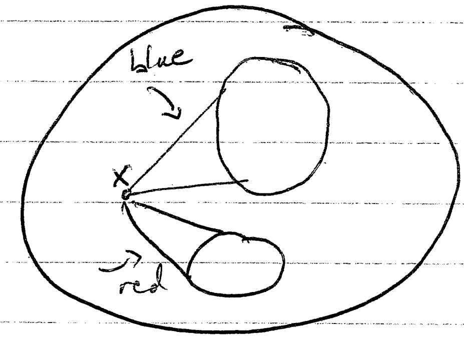
That means $100 - 39 = 61$ numbers between 1 and 100 are square-free.

#2 $r(3,4) \le r(2,4) + r(3,3)$ (justification below)

$= 4 + 6 = 10$ (explicitly calculated in class).

A brief justification of the first line:

Suppose we have a 2-colouring of all the edges of K_{10}. Fix a vertex x and consider the 9 edges from x.



By the PHP (strong version), either at least 4 of those edges are blue or at least 6 of those edges are red.

In the case we have 4 blue edges, consider the edges between the ends of the edges: they are either all red and we have a red K_4 or at least one edge is blue which means together with x we have a blue triangle.

In the case we have 6 red edges, consider the edges between the ends of the edges. Since $r(3,3)=6$ we either have a blue triangle or a red triangle. If we have a red triangle then together with x we have a red K_4

So we either have a blue triangle or a red K_4 which means $r(3,4) \leq 10$.

#3 Remember that $k\binom{n}{k} = n\binom{n-1}{k-1}$ so

$$\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} + \cdots (-1)^{n-1} n\binom{n}{n} =$$

$$n\binom{n-1}{0} - n\binom{n-1}{1} + n\binom{n-1}{2} - n\binom{n-1}{3} + \cdots (-1)^{n-1} n\binom{n-1}{n-1}$$

$$= n\left(\binom{n-1}{0} - \binom{n-1}{1} + \binom{n-1}{2} - \cdots + (-1)^{n-1}\binom{n-1}{n-1}\right)$$

$$= n \cdot 0 = 0$$

since the number of even size subsets of an $n-1$ sized set equals the number of odd sized subsets.

5. a) Suppose c_n = the number of regions on a sphere determined by n circles in general position.

How is the n^{th} circle related to the previous $n-1$? It intersects each earlier circle in two points creating a total of $2(n-1)$ arcs determined by all the intersection points. Each arc divides a previous region into 2 so

$$c_n = c_{n-1} + 2(n-1)$$

$$= c_{n-2} + 2(n-2) + 2(n-1)$$

$$\vdots$$

$$= c_1 + 2 + \cancel{2}4 + \cdots + 2(n-1)$$

$$= c_1 + n(n-1) \quad \text{and} \quad c_1 = 2$$

so $c_n = n^2 - n + 2$

b) Spheres are in general position if any three intersect in 2 points and any 4 have no common intersection. This definition is enough to guarantee that if you have n spheres in general position, the circles of intersection on any one of them are in general position. If a_n = the number of regions determined by n spheres in general position then we get

$$a_n = a_{n-1} + c_{n-1} \quad (c_n \text{ from above}).$$

$$= a_{n-2} + c_{n-2} + c_{n-1}$$

$$\vdots$$

$$= a_1 + c_1 + \cdots + c_{n-1}$$

$$= 2 + \sum_{i=1}^{n-1} (i^2 - i + 2)$$

$$= 2 + \frac{(n-1)(n^2 - 2n + 6)}{3}$$

$$= \frac{n}{3}(n^2 - 3n + 8).$$