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Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory

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We comment on some aspects of the canonical quantization of the Chern-Simons-Witten theory. We carry out explicitly the quantization on several interesting surfaces. The connection to the related two dimensional theory is illustrated from different points of view.

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1. Introduction

Recently, Witten [1] has shown that Chern-Simons gauge theories in three dimensions – CSW theories – are intimately related to rational conformal field theories in two dimensions. This leads to new insight into the structure of rational conformal field theory and might be a good starting point for a classification of these theories [2]. When the theory is studied on a three manifold of the form $\Sigma \times R$ one can interpret the non-compact direction, R, as time and use a canonical formalism to quantize it. Then one recovers the two dimensional conformal field theories in two related ways [1]. If Σ is compact, the Hilbert space $\mathcal{H}_\Sigma$ is the vector space of conformal blocks of the conformal field theory on Σ [1]. If Σ has a boundary, $\mathcal{H}_\Sigma$ is infinite dimensional and is a representation (sometimes reducible) of the chiral algebra of the conformal field theory [1].

In this paper we will study the canonical formalism in detail. There are two standard approaches to the quantization of a constrained system. In the first, one first imposes the constraints and then performs the quantization on the physical phase space. This approach was used in [1]. In section 2 we will repeat the analysis in [1] and will rephrase it in the language of Feynman path integrals. We will study the theory on different examples of Σ's. A crucial element in our analysis is an interesting map between representations and conjugacy classes of the underlying gauge group. The conjugacy class determines the holonomy of a flat connection around a source in that representation. When Σ is a disk with a source in a representation λ, the spectrum of the three dimensional theory is the corresponding representation of the loop group. For $\Sigma = T^2$ we find that for a (connected and simply connected) gauge group G, $\mathcal{H}_{T^2} \cong \frac{\Lambda^w}{W \ltimes 2k\Lambda^v}$ where Λ^w and Λ^v are the weight lattice of G and the coroot lattice, k is the coefficient of the CSW action and W is the Weyl group. In the simply laced case, this is simply $\mathcal{H}_{T^2} \cong \frac{\Lambda^w}{W \ltimes k\Lambda^R}$ where Λ^R is the root lattice of G. This space is in a natural one to one correspondence with the space of integrable representations of the affine algebra G at level k. For $\Sigma = \mathcal{A}$ an annulus we find $\mathcal{H}_\mathcal{A} \cong \oplus_\lambda \mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^*}$ where the sum is over the different integrable representations λ of the affine algebra and λ^* is the conjugate representation to λ. The study of the theory on Σ with several sources and holes leads us to a three dimensional derivation of the fusion rules in the case of $SU(2)$.

In section 3 we consider the wave functional of the states in more detail. We first find them for $\Sigma = T^2$ by an explicit evaluation of the functional integral with boundary conditions. We then adopt the second approach to canonical quantization and impose the constraints after quantization. The constraint is the statement that the wave functions are gauge invariant, i.e. they satisfy Gauss' law. Gauss' law leads to another connection between the three dimensional theory and the related two dimensional one. Using holomorphic quantization, i.e. using coherent states, we find the wave functions $\Psi(A_z)$. The wave functions on T^2 when evaluated for simple configurations, are the Weyl-Kac character formulas. More generally, the holomorphic quantization enables us to show explicitly that the wave functions are the conformal blocks of the corresponding two dimensional conformal field theory on any Σ.

In section 4 we show how the anomaly in the 2D determinant of fermions in the adjoint representation of the group leads to the peculiar shifts $k \to k+h$ and $\lambda \to \lambda + \rho$ that occur in certain formulae for expectation values of Wilson lines. More specifically, we show that by integrating out the gauge degrees of freedom one can derive an effective quantum mechanics problem for the interesting physical degrees of freedom. In section 5 we illustrate how the comments in [1] regarding the transport on moduli space can be made more specific. In particular one can obtain the Sugawara stress tensor from Chern-Simons gauge theory.

After completing most of this work, we received several preprints on this subject [3] which partially overlap with this paper.

2. First Constraining and then Quantizing

Since we will be interested in manifolds with a boundary, we start by studying the possible boundary conditions on the fields in the theory. A general criterion is to choose the boundary conditions such that there are no boundary corrections to the equations of motion. The action is given by

$$S \equiv \frac{k}{4\pi} \int_Y Tr\left(AdA + \frac{2}{3}A^3\right). \tag{2.1}$$

(Our conventions are outlined in the appendix.) Unless stated otherwise, we specialize to the case of $SU(N)$ gauge theories with the trace in the fundamental representation. For other groups, the trace should be replaced by $\langle ..., ... \rangle$ where $\langle T^a, T^b \rangle = -\delta^{ab}$. The variation of the action is

$$\delta S = \frac{k}{4\pi}\int_{\partial Y} Tr(\delta A A) + \frac{k}{2\pi}\int_Y Tr(\delta A F). \tag{2.2}$$

The boundary conditions should be such that

$$\int_{\partial Y} Tr(\delta A A) = 0. \tag{2.3}$$

One possibility is to set one of the components of A (say A_1) to zero.

In the rest of this section we will be interested in the CSW theory on $Y = \Sigma \times R$ where we interpret R as the time. If Σ has a boundary, we set the boundary conditions $A_0 = 0$ where A_0 is along the time direction. The symmetry of the theory is the group of gauge transformations which do not change the boundary conditions. These are gauge transformations which are independent of time on the boundary. Only a subgroup of this group should be viewed as a gauge symmetry. This is the set of transformations which are one at the boundary. Time independent transformations on the boundary should be viewed as a global symmetry. The spectrum is in representations of these symmetries rather than being invariant under them (as with gauge symmetries).

On a general Σ we decompose the exterior derivative $d = dt\frac{\partial}{\partial t} + \tilde{d}$ and the gauge field $A = A_0 + \tilde{A}$ into time and space components. Writing the action as

$$S = -\frac{k}{4\pi}\int_Y Tr(\tilde{A}\frac{\partial}{\partial t}\tilde{A}dt) + \frac{k}{2\pi}\int_Y Tr(A_0(\tilde{d}\tilde{A} + \tilde{A}^2)) \tag{2.4}$$

(if Y has a boundary, there is a surface term which vanishes by the $A_0 = 0$ boundary condition) we recognize that A_0 is a Lagrange multiplier which implements the constraint $\tilde{F} = \tilde{d}\tilde{A} + \tilde{A}^2 = 0$. Integrating over A_0 we obtain $\delta(\tilde{F})$. An effective action can then be derived for those $\tilde{A}$ satisfying the constraint by substituting such $\tilde{A}$ into (2.4). Because of the underlying gauge invariance, $\tilde{A}$'s which differ by gauge transformation have the same action. Hence, the classical phase space is the space of flat connections ($\tilde{F} = 0$) modulo gauge transformations [1]. We will now implement this procedure in several examples.

1. $\Sigma = D$

The simplest example of Σ with a boundary is $\Sigma = D$ a disk. This example was studied in [1]. The constraint is easily solved: $\tilde{A} = -\tilde{d}UU^{-1}$ for a single-valued map $U : D \times R \to G$.

We now change variables from $\tilde{A}$ to U in the functional integral. The measure is

$$\int D\tilde{A}\delta(\tilde{F})\cdots = \int DU\cdots \tag{2.5}$$

with the Haar measure for U. It is important that there is no Jacobian in (2.5). Since the change of variables is done separately for every time, the Jacobian, if it existed, cannot involve time derivatives and cannot change the kinetic part of the action. More precisely, equation (2.5) can be established by writing it on the lattice. Alternatively, it is straightforward to compute the Jacobian explicitly. [1] In terms of the U's the action becomes (ϕ is the angular coordinate on ∂D)

$$S = kS_C^+(U) \equiv \frac{k}{4\pi}\int_{\partial Y} Tr\Big(U^{-1}\partial_\phi UU^{-1}\partial_t U\Big)d\phi dt + \frac{k}{12\pi}\int_Y Tr(U^{-1}dU)^3 \tag{2.6}$$

which depends only on the boundary values of U. This fact has a simple explanation. We have not yet fixed the gauge in the functional integral. By a gauge transformation, we can change the value of U in the interior and therefore the effective action (2.6) should depend only on U on the boundary. So, factoring out the volume of the gauge group we recover the chiral version of the WZW path integral which we denote by CWZW. The difference between this action and the standard WZW action is in the form of the kinetic term. This "off-diagonal" term arises in light cone coordinates on the plane [4] or on any surface in complex coordinates. Here, we find it in real coordinates on $S^1 \times R$. The reason for the name CWZW is explained below.

This Lagrangian (2.6) is invariant under the transformation on the boundary

$$U(\phi, t) \to \tilde{V}(\phi)UV(t). \tag{2.7}$$

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[1] The change of variables $\tilde{A} = -\tilde{d}UU^{-1} + \epsilon$ where ϵ is a small variation transverse to the space of flat gauge fields involves a Jacobian proportional to $|det(\partial_{\bar{z}} + \partial_{\bar{z}}UU^{-1})|^2$. However, in converting $\delta(\tilde{F})$ to $\delta(\epsilon)$ we encounter the inverse of the same determinants so there is no net Jacobian in (2.5). We thank A. Kupiainen for discussions on this point.

As explained above, the invariance under $\tilde{V}$ is a global symmetry because it does not approach one in the past and in the future. Therefore, it is clear that the states in the spectrum are in a representation of the loop group LG. The gauge symmetry $V(t)$ reflects a redundancy in our parametrization of A by U and has to be fixed. Recall that if a physical Lagrangian is first order in time derivatives (see e.g. [4]), it is related to the symplectic structure ω by writing $S = \int \mathcal{A}_i(\phi) \frac{d\phi^i}{dt} dt$ where $\omega = d\mathcal{A}$. Thus, we recover the phase space as the space of based loops LG/G together with the symplectic structure

$$\omega = \frac{k}{4\pi} \oint Tr(U^{-1}\delta U) \frac{d}{d\phi}(U^{-1}\delta U). \tag{2.8}$$

Quantization of this system gives the basic representation of LG from which one obtains the chiral algebra of G current algebra [4] . In particular, the boundary values of the gauge field A_ϕ become operators satisfying the commutation relations of Kac-Moody currents. Notice that because of the kinetic term in (2.6), the spectrum is chiral (hence the name CWZW). It is easy to check that the Hamiltonian derived from (2.6) vanishes. This is a consequence of the underlying general covariance of the three dimensional theory.

If G is not simply connected, we can still write $\tilde{A} = -\tilde{d}UU^{-1}$ but U need not be single-valued. As explained in [2], there are non-trivial sectors in which U has a transition function which winds around the group. In this case, the Hilbert space has several sectors labeled by $\pi_1(G)$. The resulting chiral algebra is extended. It is not generated only by the Kac-Moody currents but also by some other holomorphic fields. Similarly, for some disconnected groups G we find the chiral algebra of an orbifold [2]. For tensor products of groups one recovers the tensor product of the chiral algebras. However, if some of the coupling constants k for some of the groups are negative, it is sometimes possible to impose different boundary conditions [2] which lead to the chiral algebra of coset models [5].

2. $\Sigma = T^2$

The simplest non-trivial example of Σ without a boundary is the torus T^2. Again, integrating over A_0, we find the constraint $\delta(\tilde{F})$ in the space directions. The most general solution of the constraint is

$$\tilde{A} = -\tilde{d}UU^{-1} + U\theta(t)U^{-1} \tag{2.9}$$

where U is single-valued and θ is a Lie algebra-valued one form which depends only on t. As elements of the Lie algebra, the two components of θ commute since $\pi_1(\Sigma) = Z \oplus Z$ is commutative. Therefore, by redefining U, they can be put in the Cartan subalgebra, $\mathcal{T}$, $\theta(t) \equiv \vec{\theta}(t) \cdot \vec{H}$ where H^i are a basis of $\mathcal{T}$. This description is redundant. A flat connection with $\vec{\theta}$ is gauge equivalent to a flat connection with $\vec{\theta} + 2\pi\vec{\alpha}$ where $\vec{\alpha}$ is a one form with values in the root lattice. Also, different values of $\vec{\theta}$ which are related by the action of the Weyl group are gauge equivalent. Therefore, the phase space is $\frac{T\times T}{W}$ where T is the maximal torus and the Weyl group W acts diagonally. As before, the change of variables from $\tilde{A}$ to U and θ has no Jacobian: $D\tilde{A}\delta(\tilde{F}) = DUD\theta$. Moreover, one can show (e.g. from a lattice definition of the measure) that the measure $D\theta$ is a linear measure on the Lie algebra. Choosing a and b cycles, defining θ_1 (θ_2) as the component of θ along the a (b) cycle, and substituting (2.9) in the action we find the effective action

$$S = -\frac{k}{4\pi}\int Tr\theta \wedge \dot{\theta} = \frac{k}{2\pi}\int \theta_1 \cdot \dot{\vec{\theta}}_2 \tag{2.10}$$

As on the disk, the fact that U decouples reflects the gauge invariance of the system. we derive the commutation relations

$$[\theta_1^i, \theta_2^j] = -i\frac{2\pi}{k}\delta^{ij} \tag{2.11}$$

Notice that as we remarked above, the measure in the functional integral is the linear measure $D\theta_1 D\theta_2$ as necessary for the interpretation of the θ's as coordinates and momenta. This system is easily quantized. If we wish to use holomorphic quantization, we choose a complex structure τ, define $\theta_z = \theta_1 + \tau\theta_2$, a holomorphic coordinate on the Jacobian, and consider wave functions to be holomorphic sections $\psi(\theta_z)$ of a line bundle over the Jacobian. In the case of a torus, a much simpler quantization is possible since a choice of a and b cycles gives a natural real polarization [2] of the phase space into coordinates, θ_2, and momenta, θ_1. Since the phase space is compact, there can only be a finite number of quantum states. More specifically, since the coordinate is compact, the momentum is quantized and the momentum eigenstates in the coordinate representation are $\langle\theta_2|\vec{\lambda}\rangle = e^{i\vec{\lambda}\cdot\vec{\theta}_2}$ where $\vec{\lambda}$ is a

[2] For a similar use of a real polarization in the quantization of the coadjoint orbit action, see [6].

weight of G. Since the momenta are also compact, and we also have to mod out by the Weyl group (recall the phase space is $\frac{T\times T}{W}$), the independent states are labeled by

$$\vec{\lambda} \in \frac{\Lambda^w}{W \ltimes k\Lambda^R} \tag{2.12}$$

where Λ^w and Λ^R are the weight and root lattices of G. More generally, for any connected and simply connected gauge group the effective action for $\vec{\theta}$ is (with the convention $\vec{\psi}^2 = 2$ for the length of the highest root)

$$S = \frac{k}{2\pi} \int \vec{\theta}_1 \cdot \dot{\vec{\theta}}_2 \tag{2.13}$$

where $\vec{\theta}$ is identified with $\vec{\theta} + 4\pi\frac{\vec{\alpha}}{\vec{\alpha}^2}$ and $\vec{\theta}$ is identified with $W(\vec{\theta})$. Hence, the states are labeled by vectors in

$$\vec{\lambda} \in \frac{\Lambda^w}{W \ltimes 2k\Lambda^v} \tag{2.14}$$

where Λ^v is the coroot lattice generated by $\frac{\vec{\alpha}}{\vec{\alpha}^2}$. Since the integrable representations of the affine algebra $\hat{G}$ are also labeled by such a coset (remember that $W \ltimes \Lambda^v$ is the affine Weyl group), the states in the Hilbert space of the quantum mechanics problem are in one to one correspondence with these integrable representations.

3. Σ= a disk with a source

A source at P in the representation $\vec{\lambda}$ can be represented [1] by a quantum mechanics problem with variables $\omega(t) \in G$ which are coupled to the gauge fields through the action

$$\int dt Tr \lambda \omega^{-1}(\partial_0 + A_0)\omega(t) \tag{2.15}$$

where $\lambda \equiv \vec{\lambda}\cdot\vec{H}$. The Lagrangian has the gauge invariance $\omega(t) \to \omega(t)h(t)$, where $h(t) \in T$ commutes with λ. Notice that this gauge invariance suffers from global anomalies unless λ is a weight. Therefore, the classical phase space is $\frac{G}{T}$ with the symplectic structure

$$Tr\lambda(\omega^{-1}\delta\omega)^2 \tag{2.16}$$

This is the standard [7] symplectic structure for the quantization of the coadjoint orbit. Integrating over A_0 we find the constraint

$$\frac{k}{2\pi}\tilde{F}(x) + \omega(t)\lambda\omega^{-1}(t)\delta^{(2)}(x - P) = 0. \tag{2.17}$$

Using polar coordinates and putting P at the origin, the solution of (2.17) is

$$\tilde{A} = -\tilde{d}\tilde{U}\tilde{U}^{-1} \tag{2.18}$$

where

$$\tilde{U} = Uexp\left(\frac{1}{k}\omega(t)\lambda\omega^{-1}(t)\phi\right) \quad , \tag{2.19}$$

U is single-valued on the disk, and $U(0,t)$ commutes with $\omega(t)\lambda\omega^{-1}(t)$. We see that the conjugacy class of the holonomy of the flat connection around P is determined by the representation at P. We thus find a map between representations of the group G and conjugacy class elements in G:

$$\vec{\lambda} \to exp(-\frac{2\pi}{k}\lambda). \tag{2.20}$$

These values of the conjugacy classes are exactly the same as those found for the holonomy – the eigenvalues of e^{θ_1} on the torus in the previous subsection. Substituting (2.18) in the action (the CSW + the coadjoint action (2.15)) and repeating the steps similar to those in the previous examples, we derive the effective action

$$S = kS_C^{+}(U) + \frac{1}{2\pi}\int_{\partial Y} Tr\lambda U^{-1}\partial_0 U \tag{2.21}$$

which again depends only on the boundary values of U. This Lagrangian is invariant under the transformation on the boundary

$$U(\phi,t) \to \tilde{V}(\phi)UV(t). \tag{2.22}$$

where $V(t)$ commutes with λ. The invariance under $V(\phi)$ guarantees that we find a representation of the loop group (more precisely, of a central extension of it). The gauge invariance under $V(t)$ shows that the phase space is LG/T with the symplectic structure derived from (2.21):

$$\frac{k}{4\pi}\oint Tr(U^{-1}\delta U)\frac{d}{d\sigma}(U^{-1}\delta U) + \frac{1}{2\pi}\oint Tr\lambda(U^{-1}\delta U)^2. \tag{2.23}$$

The quantization of this theory is known to lead to the integrable representation $\mathcal{H}_\lambda$ of the Kac-Moody algebra [8].

By examining equation (2.21) one can recover the different integrable representations. By changing variables $U \to Ue^{\phi\alpha}$ where α is a root we see explicitly that (2.21) with λ and with $\lambda + k\alpha$ are equivalent. Also, it is easy to see that the action of the Weyl group does not change the theory. We, therefore, recover the same answer we found in the quantization on T^2.

4. $\Sigma = \mathcal{A}$ annulus

By repeating the analysis before, and writing

$$\begin{aligned} \tilde{A} &= -\tilde{d}\tilde{U}\tilde{U} \\ \tilde{U} &= Ue^{\frac{\phi}{k}\lambda(t)} \end{aligned} \tag{2.24}$$

(U is a single-valued map from $Y = \Sigma \times R$ to the group) we find the effective action

$$kS_C^{+}(V_1) - kS_C^{+}(V_2) + \frac{1}{2\pi}\int Tr\lambda(t)(V_1^{-1}\dot{V}_1 - V_2^{-1}\dot{V}_2) \tag{2.25}$$

where V_1 is the value of U on one boundary and V_2 is the value on the other boundary. There is still the "gauge symmetry" (not a gauge symmetry of the underlying theory but of our parametrization) $V_i(\phi,t) \to V_i(\phi,t)\omega(t)$; $\lambda(t) \to \omega^{-1}\lambda\omega$. Also, only $e^{\frac{2\pi}{k}\lambda(t)}$ is relevant. Therefore, the phase space is $LG \times LG \times G/G$. Using the gauge invariance we can put $\lambda(t)$ in $\mathcal{T}$. There is still gauge invariance with ω in the maximal torus. We now change variables $V_1 \to V_1\omega(t)$, with ω in $\mathcal{T}$ and we see that as a dynamical variable $\omega(t)$ is conjugate to $\lambda(t)$. The quantization of this system is similar to that on the torus and hence λ is quantized. The functional integral over λ and ω becomes a finite sum over the allowed λ's, i.e. $\lambda \in \frac{\Lambda^{w}}{W \ltimes k\Lambda^{R}} = \mathcal{H}_{T^2}$. Every term (with λ independent of t) is as in the example of a disk with a source and leads to $\mathcal{H}_\lambda$ on one boundary and $\mathcal{H}_{\lambda^*}$ on the other boundary where λ^* is the conjugate representation to λ (the conjugation appears because of the sign of the action (2.25)). Therefore, the Hilbert space is

$$\mathcal{H} = \oplus_{\lambda \in \mathcal{H}_{T^2}} \mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^*} \tag{2.26}$$

It is amusing to note that this is precisely the Hilbert space of the full WZW model when both left movers and right movers are taken into account. As suggested in [2], this leads to an explicit realization of the conjecture in [9] suggesting that the left movers and the right movers have independent moduli. The left-movers of the WZW theory appear at one boundary of the annulus and the right-movers at the other boundary. More generally, by considering the CSW theory on Y which interpolates between two annuli in the past and one annulus in the future such that the inner boundaries are connected and the outer boundaries are connected (we may call such a surface a "thick pants" diagram) we find a map $\mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ where $\mathcal{H} = \oplus_i \mathcal{H}_i \otimes \mathcal{H}_{i^*}$. This is a thickened version of a three string vertex. If the inner boundaries are replaced by Wilson lines in representations i, j, k coupled together, we find a three dimensional version of the chiral vertex operators of [10] which are maps $\mathcal{H}_i \otimes \mathcal{H}_j \to \mathcal{H}_k$. In this way one can reproduce the conformal blocks for amplitudes with descendents from the CSGT.

5. Several sources and several holes on the sphere

The last two examples can easily be generalized to the case of n holes and m sources with representations $\lambda_1, ..., \lambda_m$ on S^2. The Hilbert space must be of the form

$$\mathcal{H} = \oplus_{i_1,...,i_n} V_{i_1,...,i_n,\lambda_1,...,\lambda_m} \otimes \mathcal{H}_{i_1} \otimes ... \otimes \mathcal{H}_{i_n} \tag{2.27}$$

where $V_{i_1,...,i_l}$ is the Hilbert space corresponding to quantizing l sources with representations $i_1, ..., i_l$ on S^2.

The vector space $V_{i_1,...,i_l}$ can be found as follows. Consider first the problem of quantizing the theory on R^2 with sources $i_1, ..., i_l$. The phase space is $(\frac{G}{T})^{\otimes l}$ with the symplectic structure corresponding to $i_1, ..., i_l$. When the system is considered on S^2 the phase space is reduced. As we saw earlier, a source in a representation λ determines the conjugacy class of the holonomy of the flat connection around it. This holonomy is $g_\lambda = exp(-\frac{2\pi}{k}\omega\lambda\omega^{-1})$ for some $\omega \in G/T$ A flat connection on S^2 is possible only when

$$g_{i_1} g_{i_2} \cdots g_{i_l} = 1. \tag{2.28}$$

Corresponding to this constraint, we should also mod out the phase space by the equivalence relation generated by the constraint (rigid gauge transformations). The resulting phase space is

$$\left((\frac{G}{T})^{\otimes l} / \sim \right) / G \tag{2.29}$$

Since this phase space is compact, it is clear that the Hilbert space $V_{i_1,\ldots,i_l}$ is finite [3].

In general, it is not easy to quantize this phase space. However, in the simple case of $l = 3$ and the gauge group $SU(2)$, the analysis is easy. In this case the constraint (2.28) modulo gauge transformations is satisfied only at isolated points. In the case of $SU(2)$ it is standard to label the representations by the (integral or half-integral) spin j. The representation j corresponds to the conjugacy classes of rotations by $\frac{4\pi j}{k}$. It can be represented by $exp(\frac{4\pi j}{k} J^3)$. For example, $j = \frac{k}{2}$ leads to the conjugacy class of rotation by 2π which consists of one point. This fact played an important role in the study of extended chiral algebras from the three dimensional point of view in [2]. Clearly, if $j_1 + j_2 \leq \frac{k}{2}$, equation (2.28) can be satisfied with any j_3 satisfying the triangle inequality $|j_1 - j_2| \leq j_3 \leq j_1 + j_2$. These are the standard fusion rules for coupling representations of $SU(2)$. For $j_1 + j_2 > \frac{k}{2}$ multiply both g_{j_1} and g_{j_2} in (2.28) by the central element corresponding to a 2π rotation. This transforms g_j to an element in the conjugacy class of $\frac{k}{2} - j$. Therefore, j_3 which can couple to j_1, j_2 for $j_1 + j_2 > \frac{k}{2}$ are the same as those which couple to $\frac{k}{2} - j_1$ and $\frac{k}{2} - j_2$. These j_3's can then be found using the triangle inequality $|j_1 - j_2| \leq j_3 \leq k - (j_1 + j_2)$ (where we have used $(\frac{k}{2} - j_1) + (\frac{k}{2} - j_2) < \frac{k}{2}$). For these values of j_1, j_2, j_3 the classical phase space is a point and hence the quantum Hilbert space consists of one state. Notice that this result is identical to the fusion rules as known from two dimensional considerations [11]. For other groups [4] the space of solutions of $g_{i_1} g_{i_2} g_{i_3} = 1$ modulo gauge transformations is a manifold rather than a point. Therefore, depending on k, there can be more than one state for fixed i_1, i_2, i_3. Although we cannot determine V_{i_1,i_2,i_3} in the general case, it is easy to show that $V_{i_1,i_2,i_3} \cong V_{\sigma(i_1),\sigma^{-1}(i_2),i_3}$ where $\sigma(i)$ is obtained by multiplying g_i by an element of the center. The connection between this operation and spectral flow in the Kac-Moody theory was discused in [2].

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[3] It would be interesting to see directly how quantization of such phase spaces provides invariant tensors of quantum groups $U_q(sl(n))$ for q a root of unity.

[4] We thank E. Witten for a useful discussion on the following point.

3. The physical wave function

In the previous section we saw how, in various examples, one may derive the space of states $\mathcal{H}_\Sigma$ by quantizing a reduced phase space - the moduli space of gauge fields. In this section we study the theory by quantizing on the infinite-dimensional space of all gauge fields $\mathcal{A}$. The constraint is then imposed as an operator constraint on the wavefunctions. This approach makes possible a very direct connection between wavefunctions and conformal blocks. Besides ordering the quantization and constraint procedures there are further choices that have to be made before we obtain explicit wavefunctions. In the $A_0 = 0$ gauge, the phase space is parametrized by the two components of $\tilde{A}$ which are in Σ. From the symplectic structure on $\mathcal{A}$ we obtain the equal time commutation relations:

$$[A_i^a(x), A_j^b(y)] = \frac{2\pi i}{k}\delta^{ab}\epsilon_{ij}\delta^{(2)}(x-y) \tag{3.1}$$

We must represent these commutation relations by choosing an appropriate subspace and polarization of the space of all functions on phase space. We will illustrate two approaches below. First we choose a component, say A_1, of the gauge field on Σ. Wavefunctions will then be functions of A_1 only: $\psi = \psi[A_1^a(x)]$. A very physical way to compute these wavefunctions is by direct evaluation of the path integral. As explained in [1] path integration on a three manifold whose boundary is Σ, with $A_1^a(x)$ specified on the boundary, defines a wavefunction, $\psi[A_1^a(x)]$. Below we carry this out explicitly for the torus. (The above procedure is most conveniently carried out when there is a globally defined nowhere vanishing vector field, so we restrict attention to the torus.) Unfortunately, the expressions we will find are rather singular. Also, the interpretation of the result as a wave function is complicated by ambiguities in the measure. These ambiguities are not present when the gauge group is abelian. We will then turn to an alternative quantization procedure which avoids these problems.

The alternative procedure starts with a choice of complex structure on Σ. This induces a Kähler structure on $\mathcal{A}$. Thus we may carry out Kähler quantization, which in this example is merely coherent state quantization. Once again one could evaluate the path integrals for coherent states defined by three-manifolds with boundary, but a more direct route to the same answer is to derive the action of Gauss' law on the wavefunctions $\Psi[A_z(x)]$ and demand that physical states be gauge invariant. We illustrate this procedure in a few examples.

1. The functional integral with boundary conditions on T^2

In this subsection we will evaluate the functional integral with boundary conditions. In the previous sections we argued that one has to set one of the components of the connection, say A_1 to zero at the boundary. In order to specify non-zero values of A_1 at the boundary, one needs to add a term

$$\frac{k}{4\pi}\int_{\partial Y} Tr A_1 A_2 \tag{3.2}$$

to the action (A_2 is the component of A along a direction not parallel to A_1). With this term added, the boundary term in the variation of the action is

$$\int_{\partial Y} Tr(\delta A_1 A_2) = 0. \tag{3.3}$$

Equivalently, to specify A_2, one needs to add the term

$$-\frac{k}{4\pi}\int_{\partial Y} Tr A_1 A_2. \tag{3.4}$$

This prescription can also be derived from demanding the existence of a consistent canonical formalism. There, we can view ∂Y as a surface on which we specify initial conditions for the functional integral. Then "time" is perpendicular to ∂Y. In the gauge $A_0 = 0$, we can view A_1 and A_2 as coordinates and momenta. For $A_0 = 0$ gauge, the action (2.1) does not have the standard form $\int p^i \dot{q}_i$ but instead, is of the form $\int p^i \dot{q}_i - \dot{p}^i q_i$. These two expressions differ by a surface term proportional to $p^i q_i$ which for our case is equation (3.2). It is easy to see that this prescription guarantees the proper sewing properties of amplitudes.

We can now evaluate the functional integral over a solid torus with a Wilson loop in the representation λ winding around its non-contractable cycle as a function of, say, A_2. This calculation is simple given the following observation. This functional integral is similar to the functional integral performed in section 2 for a disk with a source. If

R in that calculation is replaced by S^1, the two problems are identical. The only other difference is that there we computed it with $A_2 = 0$ (in the interpretation of this functional integral used in section 2, it was more natural to call it $A_0 = 0$). This is easily changed. We need to add the surface term $-\frac{k}{4\pi}\int TrA_1A_2$ dropped by the A_2 boundary conditions in deriving (2.4), and we need to add the term (3.4). Then, we find

$$\psi_\lambda(A_2(x)) \equiv \int_{A_2(x)} DAe^{ikS(A)}W_\lambda = \int DUe^{ikS_C^+(U,-\frac{\lambda}{k},A_2)+i\frac{1}{2\pi}\int Tr\lambda A_2} \tag{3.5}$$

where

$$\begin{aligned} kS_C^+(U,A_1,A_2) \equiv & \frac{k}{4\pi}\int_{\partial Y=T^2} Tr\Big(U^{-1}\partial_1UU^{-1}\partial_2U\Big)d\phi dt + \frac{k}{12\pi}\int_Y Tr(U^{-1}dU)^3 \\ & - \frac{k}{2\pi}\int_{\partial Y=T^2} Tr(A_1U^{-1}\partial_2U - A_2\partial_1UU^{-1} + UA_1U^{-1}A_2 - A_1A_2) \end{aligned} \tag{3.6}$$

is the gauged CWZW action. Ignoring for the moment the fact that this functional integral is divergent, we can use it to to find

$$\begin{aligned} \psi_\lambda(A_1(x)) & \equiv \int_{A_1(x)} DAe^{ikS(A)} \\ & = \int DUe^{ikS_C^+(U)+\frac{i}{2\pi}\int_{T^2} Tr\lambda(x)U^{-1}\partial_2U}\delta(A_1 + \partial_1UU^{-1} + U\frac{\lambda}{k}U^{-1}) \\ & = \int DUe^{-ikS_C^+(U)+\frac{ik}{2\pi}\int_{T^2} TrA_1(x)U^{-1}\partial_2U}\delta(A_1^U + \frac{\lambda}{k}). \end{aligned} \tag{3.7}$$

We see that $\psi_\lambda(A_1)$ has support only on A_1 which is a component of a flat connection. Also, as expected from the analysis in section 2, the holonomy of the flat connection is $e^{-\frac{2\pi}{k}\lambda}$. This fact can also be understood by evaluating the functional integral over the solid torus by first integrating over A_0. This leads to $\delta(F_{12})$. Therefore, the functional integral vanishes for every A_1 which is not a component of a flat connection. The delta function support of ψ_λ might seem disturbing. However, this is a standard phenomenon when we write a wave function in a system with constraints.

Since there is no fundamental difference between A_1 and A_2 one should expect a similar delta function support in $\psi_\lambda(A_2)$. Indeed, examining the functional integral (3.5) one easily sees that there are global anomalies unless A_2 can be written as $A_2 = g\frac{\mu}{k}g^{-1} -$

$\partial_2 g g^{-1}$ with $\mu \in \Lambda^w$. (3.7) can be evaluated on the support of the delta function. Writing $A_1 = -g\frac{\lambda}{k}g^{-1} - \partial_1 g g^{-1}$ we find

$$\psi_\lambda(A_1(x)) = e^{ikS_C^+(g)+\frac{i}{2\pi}\int_{T^2} Tr\lambda g^{-1}\partial_2 g}\psi_\lambda(A_1(x) = -\frac{\lambda}{k}) \tag{3.8}$$

Notice that although $g(x)$ is not uniquely determined by $A_1(x)$ ($g(x)$ can be multiplied on the right by any function of x_2 which commutes with λ), this ambiguity does not affect (3.8).

Clearly, the functional integral in equation (3.5) is divergent. It is divergent even after performing the standard renormalizations in field theory. This follows from the interpretation of this functional integral as a trace over the Hilbert space on the disk with a source. Since the Hamiltonian of the three dimensional theory vanishes, and the Hilbert space on the disk has an infinite number of states, (3.5) is infinite.

The interpretation of $\psi_\lambda(A_1)$ that we computed as a wave function is not as simple as it might appear. In calculating expectation values of operators one needs to integrate over A_1. Since ψ has support only on a subspace of the space of all A_1's we need to find the correct measure on that subspace. One possibility is to use the measure on this space induced from that on the full space, i.e. DA_1. A natural coordinate system on the subspace is given by the space of gauge transformations g which map A_1 to the configuration which is constant and in $\mathcal{T}$ (as in equation (3.8)). Comparing with the natural Haar measure for the space of configurations g, we find a nontrivial Jacobian $Det(\partial_1 + A_1)$. This formal expression suffers from anomalies and must be well-defined. In the next subsection we will solve this problem by using a different quantization procedure – holomorphic quantization, which, although physically less transparent has the virtue that all relevant Jacobians can be precisely defined. The holomorphic viewpoint has the added virtue that it can be generalized to all surfaces Σ.

These anomalies are not present if the gauge group is abelian. For a $U(1)$ gauge theory the action is

$$S \equiv \frac{k}{4\pi}\int AdA \tag{3.9}$$

and the gauge transformations are $A \to A - d\Lambda$ where Λ is identified with $\Lambda + 2\pi i$. Repeating the analysis in the previous sections, we learn that the allowed holonomies are $e^{\frac{2\pi i n}{k}}$ for

n an integer. The states on $\Sigma = T^2$ are labeled by the holonomy n with the state $|n\rangle$ identified with the state $|n+k\rangle$. Hence, k has to be an integer. The Verlinde operators [12] can be interpreted as Wilson lines [1]. The eigenvalues of the holonomy around the a cycle in the representation m, $V_a(m) = e^{m\oint A_1}$ are $e^{\frac{2\pi i nm}{k}}$. In the basis where V_a is diagonal, $V_b(m) = e^{m\oint A_2}$ transforms the state $|n\rangle$ to the state $|n+m\rangle$. This is precisely as expected [12]. Clearly, the interesting gauge invariant degrees of freedom are $\theta_1 = i\oint A_1$ and $\theta_2 = i\oint A_2$. Their commutation relations are $[\theta_1, \theta_2] = \frac{2\pi i}{k}$. All other degrees of freedom are trivially integrated out (this step is non-trivial in the non-abelian theory and will be carried out below).

The unitary operators T and S which implement the modular transformations $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -\frac{1}{\tau}$ can be determined by their commutation relations with θ_1 and θ_2

$$
\begin{aligned}
T\theta_1 T^{-1} &= \theta_1 \\
T\theta_2 T^{-1} &= -\theta_1 + \theta_2 \\
S\theta_1 S^{-1} &= \theta_2 \\
S\theta_2 S^{-1} &= -\theta_1 .
\end{aligned}
\tag{3.10}
$$

They are given by [13]

$$
\begin{aligned}
T &= \eta e^{i\frac{k}{4\pi}\theta_1^2} \\
S &= \eta' e^{i\frac{k}{8}(\theta_1^2+\theta_2^2)}
\end{aligned}
\tag{3.11}
$$

where the phases η and η' cannot be determined by (3.10) but can be partially fixed by the relations of the modular group. From the expression for T we deduce that the conformal dimensions satisfy $\Delta_n mod 1 = \frac{n^2}{2k} mod 1$. We see that the identification of the state $|n\rangle$ with the state $|n+k\rangle$ is consistent with the expression for Δ only if k is an even integer.

These results are easily generalized for an abelian gauge group $\frac{R^d}{\Gamma}$ where Γ is a d dimensional lattice. We can represent it by the action

$$
S \equiv \frac{k_{ij}}{4\pi} \int A^i dA^j \tag{3.12}
$$

where k_{ij} is the metric of the lattice and the gauge symmetry is $A^i \rightarrow A^i - d\Lambda^i$ and Λ^i is identified with $\Lambda^i + 2\pi i$. The quantization on the torus leads to states labeled by a vector of integers n_i $(i = 1, ..., d)$. Because of the gauge invariance, the state $|n_i\rangle$ is identified with the state $|n_i + k_{ji}\rangle$ for every i and j. Hence, k_{ij} have to be integers (the lattice is an integral lattice). This is analogous to the conclusion that k is quantized in the $U(1)$ example. There, we saw that good behavior under T leads to the requirement that k is even. In this case we learn that for consistency the lattice must be even. Hence, Λ has to be an even integral lattice.

2. The wave function in holomorphic quantization

In this subsection we will construct the wavefunction in holomorphic quantization. Instead of direct evaluation of the path integral used in the previous subsection we will first quantize the theory and then impose Gauss' law. Gauss' law is the statement that the physical wave functions are gauge invariant.

Consider first a surface Σ without sources. We will use axial gauge $A_0 = 0$. Picking a complex structure, we can split the two-dimensional connection $\tilde{A}$ into A_z and $A_{\bar{z}}$.

The commutation relations are then

$$[A_z^a(z_1), A_{\bar{z}}^b(z_2)] = \frac{2\pi i}{k}\delta^{ab}\delta^{(2)}(z_1 - z_2) \tag{3.13}$$

(here $\delta^{(2)}(z)$ is normalized such that it is integrated to 1 with the measure d^2z) and wavefunctions are in the "coherent state representation," that is, they are taken to be holomorphic functions of A_z. The inner product of two wave functions is

$$\langle\Psi_1|\Psi_2\rangle = \int DA_z^a(x)D(A_z^a(x)^*)e^{-\frac{ik}{2\pi}\int TrAA^*}(\Psi_1[A_z])^*\Psi_2[A_z] \tag{3.14}$$

The exponential prefactor is the Kahler potential, as is standard in holomorphic quantization.

The physical Hilbert space $\mathcal{H}_\Sigma$ is the gauge invariant subspace of the infinite dimensional Hilbert space of all functionals of A_z. To find $\mathcal{H}_\Sigma$, we should first construct the operator which generates gauge transformations.

From the equation of motion of A_0 it is clear that the generator of Gauss' law, for an infinitesimal gauge transformation by $g = e^{\epsilon(x)}$ is given by:

$$u(\epsilon) = \frac{k}{2\pi}\int Tr\epsilon F \tag{3.15}$$

where $F_{z,\bar{z}}(x) = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}]$. In the quantum theory where A_z and $A_{\bar{z}}$ do not commute, the exponentiation of F is not trivial. The unitary operator $U(g(x))$ which implements a finite, space dependent, gauge transformation $g(x)$ can be constructed by imposing the following requirements:

$$U(g)AU^\dagger(g) = A^g = gAg^{-1} - dgg^{-1} \tag{3.16}$$

$$U(g_2)U(g_1) = U(g_1 g_2) \tag{3.17}$$

Equation (3.17) expresses the fact that $U(g)$ is a representation of the gauge group (no local gauge anomalies). On wave functionals of A_z (3.16) is satisfied for

$$U(g)\Psi(A_z) = exp\left\{-\frac{ik}{2\pi}\int_\Sigma d^2z Tr(A_z g^{-1}\partial_{\bar{z}} g) + ikf(g)\right\}\Psi(A_z^g) \tag{3.18}$$

where $f(g)$ is an A_z-independent phase. Equation (3.17) leads to

$$f(g_1 g_2) = f(g_1) + f(g_2) + \frac{1}{2\pi} Tr(\partial g_2 g_2^{-1} g_1^{-1} \partial_{\bar{z}} g_2) \tag{3.19}$$

Using the (PW) identity [14] it is easy to see that for gauge transformations deformable to the identity, the unique solution of (3.19) is:

$$(U(g)\Psi)(A_z) = e^{ikS^+(g,A_z,A_{\bar{z}}=0)}\Psi(A_z^g) \tag{3.20}$$

If external sources in representations λ_i at the points P_i are present, the wave function before imposing the constraint is an A_z dependent vector in $\otimes_i R_{\lambda_i}$. Gauge transformations are implemented by

$$(U(g)\vec{\Psi})(A_z) = e^{ikS^+(g,A_z,A_{\bar{z}}=0)} \otimes_i \rho_{\lambda_i}(g^{-1}(P_i))\vec{\Psi}(A_z^g) \tag{3.21}$$

where $\rho_{\lambda_i}(g)$ is the group element g in the representation λ_i.

A gauge invariant wave function Ψ is obtained by applying the projection operator $\int DgU(g)$ on any test function Ψ_0. Consider a test function of the form $\Psi_0^{(\bar{J})}(A) = e^{\frac{ik}{2\pi}\int Tr\bar{J}A}$. It leads to

$$\Psi^{(\bar{J})}(A) = e^{\frac{ik}{2\pi}\int Tr\bar{J}A}\int Dg e^{ikS^+(g,A_z,A_{\bar{z}}=\bar{J})} \tag{3.22}$$

This is the functional integral one has to calculate to find the partition function of the WZW theory in the presence of gauge fields coupled to the currents. The calculation is simpler for $\Psi^J(A=0)$ where it can be carried out in perturbation theory in J. Then one can use Kac-Moody Ward identities which were studied in [15] [16] [17] to learn that by varying $\bar{J}$, the independent wave functions $\Psi(A=0)$ are the different conformal blocks on Σ. We will make this argument more explicit in some examples.

We first illustrate the above argument for the case of a sphere with sources. In this case the physical wavefunctions are given by

$$\vec{\Psi}(A_z) = \int Dg e^{ikS^+(g,A_z,A_{\bar{z}}=0)} \otimes_i \rho_{\lambda_i}(g^{-1}(P_i))\vec{\Psi}_0(A_z^g) \tag{3.23}$$

From this formula we can understand why a basis of wavefunctions is in natural correspondece with the conformal blocks in two ways. For almost all A_z we may write $A_z = -\partial hh^{-1}$ [18] [19], the value of the wavefunction at A_z is determined from its value at $A=0$ by the Ward identities. At $A=0$ the testfunction Ψ_0 becomes an arbitrary functional of the holomorphic Kac-Moody currents, $\Psi_0(-\partial gg^{-1}) = \Psi_0(J)$, so that the integral (3.23) is of the form

$$\Psi^{m_1\ldots m_n}(A_z=0;\bar{z}_i) = \sum_{\vec{p}} \bar{\mathcal{F}}_{\vec{p}}^{m_1,\ldots m_n}(\bar{z}_1,\ldots \bar{z}_n)\mathcal{F}_{\vec{p}}^{m_1',\ldots m_n'}(z_1,\ldots z_n)\mathcal{O}_{m_1'\ldots m_n'}(z_1,\ldots z_n) \tag{3.24}$$

where $\mathcal{O}$ depends holomorphically on z_i and may be varied by varying Ψ_0. By looking at the first few terms in an expansion of Ψ_0 it is easy to see that, if we postulate that the space of states varies antiholomorphically with the moduli $\bar{z}_i$ then the space of quantum states is naturally isomorphic to the space of conformal blocks. There is an alternative point of view on how to use (3.23) . The path integral (3.23) is manifestly holomorphic in A_z, but the integral might not make sense, or might vanish. Instead one could use the relation $A_z = -\partial hh^{-1}$ to *define* the value $\vec{\Psi}[A_z]$ for nonvanishing A_z in terms of an arbitrary tensor $\vec{\Psi} \in \otimes_i R_i$. That is, we define $\vec{\Psi}[A_z] \equiv e^{-ikS(h)} \otimes_i \rho_{j_i}(h(P_i))\vec{\Psi}$. Invariance under Gauss' law for global gauge transformations forces Ψ to be an invariant tensor, but once this condition is met, $\vec{\Psi}[A_z]$ formally satisfies the Gauss law constraint for all gauge transformations. However, we cannot always gauge A_z to zero via $A_z = -\partial hh^{-1}$.

There is a codimension one subspace of gauge fields A_z which cannot be gauged to zero, and for which the above definition will not work. Demanding that the wavefunction Ψ is nevertheless a holomorphic function of A_z leads once again to the correspondence of states and conformal blocks. This latter point of view has been emphasized by Felder, Gawedzki, and Kupiainen [20].

4. An Effective Quantum Mechanics Problem

In the previous sections we saw that the flat connection around a source λ has holonomy in the conjugacy class of $e^{-2\pi\frac{\lambda}{k}}$. Interpreting the Verlinde operators [12] as Wilson loops as in reference [1], we expect that the holonomy measured in the quantum theory is in the conjugacy class of $e^{-2\pi\frac{\lambda+\rho}{k+h}}$ (ρ and h are half the sum of the positive roots and the dual Coxeter number respectively). In this section we will see how to understand the shifts $k \to k+h$ and $\lambda \to \lambda+\rho$ in terms of an effective quantum mechanics problem. The effective quantum mechanics problem is defined by splitting the coordinates A_z into gauge and moduli degrees of freedom. The essential point is that in the evaluation of expectation values one must integrate out the gauge degrees of freedom. In so doing one encounters nontrivial Jacobians. The source of the shift is the anomaly in these Jacobians. There are some interesting analogies [5] between the present calculation and the work of I. Frenkel [21].

We specialize our discussion to the case of $\Sigma = T^2$. Conventions for theta functions and Weyl-Kac characters may be found in the appendix. As explained above, the physical wave functions are gauge invariant

$$\Psi(A_z) = (U(g)\Psi)(A_z) = e^{ikS^+(g;A,\bar{A}=0)}\Psi(A_z^g) \tag{4.1}$$

Moreover, if we restrict the physical, gauge invariant wavefunctions to $A_z = a_z$, where a_z is a constant in $\mathcal{T}$, then Gauss' law tells us we have level k theta functions. More specifically, defining

$$\vec{a} \equiv \frac{i\pi}{Im\tau}\vec{u}^* \tag{4.2}$$

[5] We thank E. Verlinde for bringing Frenkel's paper to our attention.

a basis for physical wavefunctions is given by

$$\Psi_\lambda[\vec{a}] \equiv e^{\frac{k\pi}{2Im\tau}(\vec{u}^*)^2}\bar{\chi}_\lambda(\bar{\tau},\vec{u}^*) \tag{4.3}$$

for distinct $\lambda \in \Lambda^w/W \ltimes k\Lambda^R$. (These span the space of holomorphic sections, invariant under the action of the Weyl group, of the k'th power of the line bundle $Det\bar{\partial}$ over the moduli space of flat connections, in accordance with the general arguments in [1] .) One might be tempted to ignore the A_z configurations which are not constant and in $\mathcal{T}$ and consider the wave functions as functions of $\vec{a}$. Then it is easy to transform to coordinate representation – wave functions of a_1 and find that they are delta functions with support on $a_1 = \frac{\lambda}{k}$ as found above. However, in the previous discussion the wave function was more complicated than that and involved an infinite factor which has to be handled with care. The corresponding statement in the holomorphic quantization is that we should not just set $A_z = a_z$ but we should integrate out the other modes to find an effective quantum mechanics problem for a_z.

We do that by considering the inner product of two wave functions. Note that on the torus for all A_z except in a subset of codimension one there exists a gauge transformation so that $(A_z)^g = a_z$ is constant and in $\mathcal{T}$ where $g : \Sigma \to G_C$ and G_C is the complexification of the group G. (Therefore, $A_{\bar{z}}$ is gauge transformed to a constant by g^*, so we are not saying that the field is pure gauge.) Since the gauge transformation law follows from a differential equation which can be analytically continued, we can use it for gauge transformations in G_C. Thus we know the value of the wavefunction for all $A_z(x)$ in terms of a_z:

$$\Psi_\lambda[A_z(x)] = exp\left[-ikS^-(g) - \frac{ik}{2\pi}\int Tra\bar{\partial}gg^{-1}\right]\Psi_\lambda[a] \tag{4.4}$$

Recalling the inner product (3.14) and using (4.4) and the PW formula, the overlap is

$$\langle\Psi_1|\Psi_2\rangle = \int DADA^* e^{-ikS^-(gg^*;a,-a^*)-\frac{ik}{2\pi}\int Traa^*}(\Psi_{\lambda_1}(a))^*\Psi_{\lambda_2}(a) \tag{4.5}$$

Now we change the integration variables from A_z to g and account for the Jacobian $Det(\partial + A)$ as in [22] to get

$$\int dada^* e^{-\frac{ik}{2\pi}\int Traa^*}\left[e^{\frac{i2h}{4\pi}\int Tr(a-a^*)^2}|\Pi(\tau,u)|^4\right] (\Psi_{\lambda_1}(a))^*\Psi_{\lambda_2}(a)\int Dg e^{-i(k+2h)S^-(gg^*;a,-a^*)} \tag{4.6}$$

where u is related to a as in (4.2) and h is the dual Coxeter number. Now again we can appeal to [22] to find the value of the path integral over g, it is just

$$e^{-\frac{ih}{4\pi}\int Tr(a-a^*)^2}\frac{1}{|\Pi(\tau,u)|^2} \tag{4.7}$$

Thus, if we take a basis of the physical wavefunctions to be (on the constant $\mathcal{T}$) the WK characters then all the Π's cancel and we are left with ordinary theta functions, but at level $k+h$ and at weight $\lambda+\rho$. More precisely, if we define an effective wavefunction

$$\Psi^{eff}_{\lambda}(\vec{a}) \equiv e^{\frac{i(k+h)}{4\pi}\int Tra^2}\bar{\Theta}_{\overline{\lambda+\rho,k+h}}(\bar{\tau},\frac{-iIm\tau}{\pi}\vec{a}) \tag{4.8}$$

then we get for the overlap

$$\int dada^* e^{\frac{-i(k+h)}{2\pi}\int Traa^*}(\Psi^{eff}_{\lambda_1}(a))^*(\Psi^{eff}_{\lambda_2}(a)) \propto \delta_{\lambda_1,\lambda_2}. \tag{4.9}$$

Notice that in this expression the region of integration is compact.

We may interpret the above calculation as follows. We have an effective quantum mechanics problem from integrating out the degrees of freedom. This effective system is related to the original one, obtained by restriction to $\mathcal{T}$, by the famous shifts $k \to k+h$, $\lambda \to \lambda+\rho$. Now one can define a corresponding effective coordinate representation relating a_1 to a_z etc. according to the above conventions. The kernel for passing from coordinate wavefunctions to holomorphic wavefunctions is easily found to be

$$K(a_z,a_1) = exp\left[\frac{-ik\bar{\tau}}{4\pi}\vec{a}_1^2 + \frac{kIm\tau}{\pi}\vec{a}_1\cdot\vec{a}_z - \frac{kIm\tau}{2\pi}(\vec{a}_z)^2\right] \tag{4.10}$$

So defining the periodic delta function

$$\delta^P(\vec{a}_1-\vec{\ell}) \equiv \sum_{\alpha\in\Lambda^R}\delta(\vec{a}_1-\vec{\ell}-2\pi\alpha) \tag{4.11}$$

we get

$$\Psi^{eff}_{\lambda}(\vec{a}_1) = \sum_{w\in W}(-1)^w\delta^P(\vec{a}_1 - 2\pi\frac{w(\lambda+\rho)}{k+h}) \tag{4.12}$$

From (4.9) it is easy to find the commutators of the dynamical variables a and a^*. Using the relation of these variables to the real a_1 and a_2 variables one finds that in the effective quantum mechanics problem

$$[a_1^i,a_2^j] = \frac{2\pi i}{k+h}\delta^{ij}. \tag{4.13}$$

From this effective quantum mechanics problem, we see that rather than labeling the spectrum by the Weyl alcove $\frac{\Lambda^w}{W \ltimes k\Lambda^R}$, we can label it by the interior of the dilated alcove $\frac{\Lambda^w}{W \ltimes (k+h)\Lambda^R}$. Such a parametrization is standard in the study of Kac-Moody algebras.

Strictly speaking, to justify the above interpretation we should show that the same phenomenon persists when evaluating matrix elements of nontrivial gauge-invariant operators. The analysis trivially generalizes to the case of operators whose holomorphic kernels $\mathcal{O}(A_z, B_z^*)$ satisfy the invariance law $\mathcal{O}(A_z^g, B_z^*) = \mathcal{O}(A_z, B_z^*)$ etc. but unfortunately this does not include all gauge invariant operators. We will assume the same phenomenon persists for all gauge invariant operators and proceed, but the demonstration that this is the case remains an interesting open problem. Given this assumption, we can evaluate different operators as we did in the abelian case above.

In terms of the effective quantum mechanics problem, various operators are very simple. For instance, the Verlinde operators are $V_\mu(C_a) = Tr_\mu P e^{\oint \vec{a}_1 \cdot H}$ and $V_\mu(C_b) = Tr_\mu P e^{\oint \vec{a}_2 \cdot H}$. As in the abelian theory, the unitary operators T and S which implement the modular transformations $\tau \to \tau + 1$ and $\tau \to -\frac{1}{\tau}$ can be determined by their commutation relations with a_1 and a_2

$$
\begin{aligned}
&T a_1 T^{-1} = a_1 \\
&T a_2 T^{-1} = -a_1 + a_2 \\
&S a_1 S^{-1} = a_2 \\
&S a_2 S^{-1} = -a_1 .
\end{aligned}
\tag{4.14}
$$

They are given by

$$
\begin{aligned}
&T = \eta e^{i \frac{k}{4\pi} a_1^2} \\
&S = \eta' e^{i \frac{k}{8} (a_1^2 + a_2^2)}
\end{aligned}
\tag{4.15}
$$

The existence of the phase η is related to the central charge c. From the expression for T we find the conformal dimensions modulo an integer to be $\frac{\lambda(\lambda+2\rho)}{2(k+h)}$.

5. Transport on Moduli Space

In [1] it was argued that since the symplectic structure on phase space may be defined without reference to a complex structure on the Riemann surface Σ, when one does use the complex structure to quantize, the resulting space of quantum states must define a flat vector bundle on the moduli space of curves. In this section we indicate how the parallel transport on moduli space may be seen quite explicitly using the previous formalism. We consider two explicit cases, namely, the sphere with sources and the torus.

Consider first the case of the sphere with sources. As we have seen, from Gauss' law we need only discuss how the states $\vec{\psi}[A = 0; P_i]$ depend on the positions of the punctures. Neither the phase space nor the polarization nor the symplectic structure changes as we move the points. The operator which implements the Gauss law, namely, $u(\epsilon) = \frac{k}{2\pi}\int Tr\epsilon F + \sum \rho_i(T^a)\epsilon^a(P_i)$ does change: $\overline{\partial}_i u(\epsilon) = \overline{\partial}\epsilon^a(P_i)\rho_i(T^a)$ and this is the only source of the change in the physical states. Precisely this change may be effected by commutation with the operator $O = \rho_i(T^a)A^a_{\bar{z}}(P_i)$, namely $[O, u(\epsilon)] = \overline{\partial}_i u(\epsilon)$ and hence physical states must satisfy the differential equation

$$
\begin{aligned}
\overline{\partial}_i\vec{\psi}[A; P_i] &= \rho_i(T^a)A^a_{\bar{z}}(P_i)\vec{\psi}[A, P_i] \\
&= \rho_i(T^a)A^a_{\bar{z}}(P_i)\int Dg e^{ik\left(S - \frac{1}{2\pi}\int TrAg^{-1}\overline{\partial}g\right)} \otimes_i g^{-1}(z_i, \bar{z}_i)\vec{\psi}_0[A^g]
\end{aligned}
\tag{5.1}
$$

It is sufficient to consider $\vec{\psi}_0$ to be constant. Evaluating the differential equation at $A = 0$ we find

$$
\begin{aligned}
\overline{\partial}_i\vec{\psi}[A; P_i]|_{A=0} &= \rho_i(T^a)A^a_{\bar{z}}(P_i)\int Dg e^{ik\left(S - \frac{1}{2\pi}\int TrAg^{-1}\overline{\partial}g\right)} \otimes_i \rho_i\left(g^{-1}(z_i, \bar{z}_i)\right)\vec{\psi}_0|_{A=0} \\
&= \int Dg e^{ikS}\frac{1}{k}\bar{J}^a(z_i)\rho_i(T^a)\rho_i(g^{-1}(z_i, \bar{z}_i)) \otimes_{i\neq j} \rho_j(g^{-1}(P_j))\vec{\psi}_0
\end{aligned}
\tag{5.2}
$$

We must define the singular product of operators at P_i. We do this by point splitting, then making an appropriate subtraction, which will be uniquely determined from self-consistency. From conformal field theory we have the well known operator product relation [15]

$$
\bar{J}^a(\bar{\varsigma})\rho_i(T^a)g^{-1}(z_i, \bar{z}_i) = \frac{C_i}{\bar{\varsigma} - \bar{z}_i} - \frac{1}{2}(k + h)\overline{\partial}_i g^{-1}(z_i, \bar{z}_i) + O(\varsigma - z_i)
\tag{5.3}
$$

where h is the dual Coxeter number and $C_i = C_2(V^{j_i})$ is the Casimir of the representation V^{j_i}. Thus it is clear that we must define the singular product of operators by

$$
\begin{aligned}
:\rho_i(T^a)\bar{J}^a(\bar{z}_i)\rho_i(g^{-1}(z_i,\bar{z}_i)):\equiv \lim_{\varsigma\to z}\Big[\rho_i(T^a)\bar{J}^a(\varsigma)\rho_i\Big(g^{-1}(z_i,\bar{z}_i)\Big) \\
-\frac{C_i}{\varsigma-\bar{z}_i}-\frac{h}{k}\bar{\partial}_i g^{-1}(z_i,\bar{z}_i)\Big]
\end{aligned}
\tag{5.4}
$$

Plugging in this definition and using the Kac-Moody Ward identities for $\bar{J}$ we find that physical states satisfy the Knizhnik-Zamolodchikov equations

$$
(k+h)\overline{\partial}_i\vec{\psi}[0;P_i]=\sum_{j\neq i}\frac{\rho_i(T^a)\rho_j(T^a)}{\bar{z}_i-\bar{z}_j}\vec{\psi}[0;P_i]
\tag{5.5}
$$

(Readers familiar with [15] will recognize that this is essentially the original argument of Knizhnik-Zamolodchikov, the only (slight) novelty here is its interpretation from the three-dimensional viewpoint.)

We may understand parallel transport at genus one in a slightly different way. In this case as we change the complex structure Gauss' law does not change, rather, the complex structure on the space of gauge fields changes. Note that the pair $(A_z,(\bar{\tau}-\tau)A_{\bar{z}})$ for different values of τ must be related by a canonical transformation. One may check that this transformation is simply given by $A_z\to UA_zU^{-1}$ where

$$
\begin{aligned}
U &= e^{i(\delta\tau\mathcal{O}+\delta\bar{\tau}\bar{\mathcal{O}})}\\
\mathcal{O} &= \frac{k}{8\pi}\Big[\int\psi^{zz}A_z^2+\int A_zA_{\bar{z}}+A_{\bar{z}}A_z\Big]\\
\bar{\mathcal{O}} &= \frac{k}{8\pi}\int\bar{\psi}^{\bar{z}\bar{z}}A_{\bar{z}}^2
\end{aligned}
\tag{5.6}
$$

where $\bar{\psi}^{\bar{z}\bar{z}}$ is the quadratic differential dual to the tangent vector $\partial/\partial\bar{\tau}$. As in the case of the sphere with punctures, operating on the wavefunctions written as an average over gauge transformations the operator $\bar{\mathcal{O}}$ differentiates the wavefunction twice bringing down two factors of the current: $J^a(z)J^a(z)$. This singular operator must be defined by an appropriate subtraction. Again, from the paper of Knizhnik and Zamolodchikov we know that the $\mathcal{O}(1)$ term in the operator product expansion is proportional to the standard Sugawara stress energy tensor, which is known from CFT to give the correct parallel transport on moduli space. As before the proper definition of the product involves a nontrivial subtraction of the $\mathcal{O}(1)$ piece, which may again be determined from self-consistency.

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Appendix A. conventions

Generators of the lie algebra T^a are antihermitian $(T^a)^* = -T^a$ and we have

$$Tr(T^aT^b) = -x\delta^{ab}$$

x depends on the representation and on the normalization of the Killing metric. $l_r = \frac{x}{\vec{\psi}^2}$ where $\vec{\psi}$ is the highest root, depends only on the representation. It is standard to normalize $\vec{\psi}^2 = 2$.

Gauge transformations: $A = A_i^a T^a dx^i$ is the gauge field, which transforms as:

$$A^g \equiv gAg^{-1} - dgg^{-1} \tag{A.1}$$

so $(A^g)^h = A^{hg}$.

CSW action:

$$\frac{k}{8\pi}\vec{\psi}^2 \int_Y \langle A, dA + \frac{1}{3}[A, A]\rangle$$

with $\langle T^a, T^b\rangle = -\delta^{ab}$. In the normalization $\vec{\psi}^2 = 2$ we can write for $SU(N)$,

$$\frac{k}{4\pi}\int_Y Tr(AdA + \frac{2}{3}A^3)$$

with the trace in the fundamental representation ($TrT^aT^b = -\delta^{ab}$). Every trace in this paper is a trace in the fundamental representation of $SU(N)$.

WZW actions: We use a metric to split the exterior derivative into $d = \partial + \overline{\partial}$. Then

$$S^{\pm}(g) = \frac{1}{4\pi}\int_{\Sigma} Tr g^{-1}\partial g g^{-1}\overline{\partial} g \pm \frac{1}{12\pi}\int_{B} Tr(g^{-1}dg)^3$$

so the PW formulae are

$$\begin{aligned} S^{+}(gh) &= S^{+}(g) + S^{+}(h) + \frac{1}{2\pi}\int Tr\partial h h^{-1} g^{-1}\overline{\partial} g \\ S^{-}(gh) &= S^{-}(g) + S^{-}(h) + \frac{1}{2\pi}\int Tr g^{-1}\partial g\overline{\partial} h h^{-1} \end{aligned} \tag{A.2}$$

Therefore,

$$\begin{aligned} S^{+}(h_1(z) g h_2(\bar{z})) &= S^{+}(g) \\ S^{-}(h_1(\bar{z}) g h_2(z)) &= S^{-}(g) \end{aligned} \tag{A.3}$$

Gauged WZW action:

$$\begin{aligned} S^{+}(g; A, \bar{A}) &\equiv S^{+}(g) - \frac{1}{2\pi}\int Tr\Big[A g^{-1}\overline{\partial} g + \bar{A}\partial g g^{-1} + g A g^{-1}\bar{A} - A\bar{A}\Big] \\ S^{-}(g; A, \bar{A}) &\equiv S^{-}(g) + \frac{1}{2\pi}\int Tr\Big[A\overline{\partial} g g^{-1} + \bar{A} g^{-1}\partial g - g^{-1} A g\bar{A} + A\bar{A}\Big] \end{aligned} \tag{A.4}$$

which is invariant under $g \to hgh^{-1}$ with $A, \bar{A}$ gauge transformed as in (A.1), $A \to A^h$ etc.

CWZW action on $S^1 \times R$ with obvious coordinates on the boundary $t = x^0$ and ϕ.

$$S_C^{\pm}(g) = \frac{1}{4\pi}\int_{\Sigma} Tr g^{-1}\partial_{\phi} g g^{-1}\partial_0 g \pm \frac{1}{12\pi}\int_{B} Tr(g^{-1}dg)^3$$

Gauged CWZW action:

$$\begin{aligned} S_C^{+}(g; A_{\phi}, A_0) &\equiv S_C^{+}(g) - \frac{1}{2\pi}\int Tr\Big[A_{\phi} g^{-1}\partial_0 g - A_0\partial_{\phi} g g^{-1} + g A_{\phi} g^{-1} A_0 - A_{\phi} A_0\Big] \\ S_C^{-}(g; A_{\phi}, A_0) &\equiv S_C^{-}(g) + \frac{1}{2\pi}\int Tr\Big[A_{\phi}\partial_0 g g^{-1} - A_0 g^{-1}\partial_{\phi} g - g^{-1} A_{\phi} g A_0 + A_{\phi} A_0\Big] \end{aligned} \tag{A.5}$$

Conventions for the torus are

$$\begin{aligned}
0 \leq & \sigma^1, \sigma^2 \leq 1 \\
z &= \sigma^1 + \tau\sigma^2 \\
\bar{z} &= \sigma^1 + \bar{\tau}\sigma^2 \\
A_z &\equiv \frac{A_2 - \bar{\tau}A_1}{\tau - \bar{\tau}} \\
A_{\bar{z}} &\equiv \frac{\tau A_1 - A_2}{\tau - \bar{\tau}} \\
[A^a_{\bar{z}}(x), A^b_z(y)] &= \frac{\pi}{kIm\tau}\delta^{ab}\delta^{(2)}(x-y) \\
dzd\bar{z} &= -2iIm\tau d^2\sigma
\end{aligned} \tag{A.6}$$

Level k theta functions for Λ^R the root lattice of some Lie algebra and a weight $\gamma \in \Lambda^w$ are defined as

$$\begin{aligned}
\Theta_{\gamma,k}(\tau,u) &\equiv \sum_{\alpha\in\Lambda^R} e^{i\pi k\tau(\alpha+\frac{\gamma}{k})^2+2\pi ik(\alpha+\frac{\gamma}{k})\cdot u} \\
\bar{\Theta}_{\gamma,k}(\bar{\tau},u) &\equiv \sum_{\alpha\in\Lambda^R} e^{-i\pi k\bar{\tau}(\alpha+\frac{\gamma}{k})^2-2\pi ik(\alpha+\frac{\gamma}{k})\cdot u}
\end{aligned} \tag{A.7}$$

As γ runs over $\Lambda^w/k\Lambda^R$ these functions span the space of functions with the periodicity properties

$$\begin{aligned}
f(\tau, u+\beta) &= f(\tau,u) \\
f(\tau, u+\beta\tau) &= e^{-i\pi k\tau\vec{\beta}^2-2\pi ik\vec{\beta}\cdot\vec{u}}f(\tau,u)
\end{aligned} \tag{A.8}$$

for $\beta \in \Lambda^R$. We will use the combinations even and odd under the action of the Weyl group:

$$\begin{aligned}
\Theta^+_{\gamma,k} &= \sum_{w\in W}\Theta_{w(\gamma),k} \\
\Theta^-_{\gamma,k} &= \sum_{w\in W}(-1)^w\Theta_{w(\gamma),k}
\end{aligned} \tag{A.9}$$

So the Weyl-Kac characters:

$$\chi_{\lambda,k}(\tau,u) = \frac{\Theta^-_{\lambda+\rho,k+h}(\tau,u)}{\Pi(\tau,u)} \tag{A.10}$$

can be expanded in terms of $\Theta^+_{\lambda,k}$ the coefficients being parafermion partition functions for G/T. Finally note that it is

$$exp(\frac{\pi k\vec{u}^2}{2Im\tau})\Theta_{\gamma,k}(\tau,u) \tag{A.11}$$

which are orthogonal in the natural inner product on $T = \mathcal{T}/\Lambda^R$, ($\mathcal{T}$ is the Cartan subalgebra) namely

$$\langle f(u)|g(u)\rangle \equiv \int_{T=\mathcal{T}/\Lambda^R} dudu^* e^{-\frac{\pi k \vec{u}\cdot\vec{u}^*}{Im\tau}} (f(u))^* g(u) \tag{A.12}$$

References

[1] E. Witten, "Quantum Field Theory and the Jones polynomial," IAS-HEP-88/33, to appear in Comm. Math. Phys.

[2] G. Moore and N. Seiberg, "Taming the Conformal Zoo," to appear in Phys. Lett.

[3] M. Bos and V.P. Nair, "$U(1)$ Chern-Simons and $c = 1$ Conformal Blocks," Print-89-0118; Y. Hosotani, "On the Gauge Invariance in the Chern-Simons Theory on the Torus," IAS-HEP-89/8; G.V. Dunne, R. Jackiw and C.A. Trugenberger, "Chern-Simons in the Schrodinger Representation," MIT-CTP-1711 (1989); K. Gawedzki, "WZW Conformal Field Theory," IHES/P/89/06.

[4] E. Witten, Comm. Math. Phys. **92** (1984) 455.

[5] P. Goddard, A. Kent, and D. Olive, Comm. Math. Phys. **103**(1986)105.

[6] A. Alekseev, L. Faddeev, S. Shatashvili, *J. Geom. and Phys.*, in press.

[7] A.A. Kirillov, *Elements of the Theory of Representations* (Springer-Verlag, 1979).

[8] N. Pressley and G. Segal, *Loop Groups* (Oxford Univ. Press 1986).

[9] E. Witten, Phys. Rev. Lett. **61** (1988) 670.

[10] B. Schroer, Nucl. Phys. **B295** (1988) 4; "Algebraic Aspects of Non-Perturbative Quantum Field Theories," Como lectures; K.-H. Rehren, Comm. Math. Phys. **116** (1988) 675; J. Frohlich, "Statistics of Fields, the Yang-Baxter Equation, and the Theory of Knots and Links," lectures at Cargese 1987, to appear in *Nonperturbative Quantum Field Theory*, Plenum; G. Felder and J. Frohlich, unpublished lecture notes; K.-H. Rehren and B. Schroer, "Einstein Causality and Artin braids," FU preprint 88-0439; A. Tsuchiya and Y. Kanie, "Vertex operators in the conformal field theory on P1 and monodromy representations of the braid group", in *Conformal Field Theory and Solvable Lattice Models,* Advanced Studies in Pure Mathematics, **16** (1988) 297; Lett. Math. Phys. **13** (1987) 303; G. Moore and N. Seiberg, Phys. Lett. **212B** (1988) 451; Nucl. Phys. **B313** (1989) 16; "Classical and Quantum Conformal Field Theory," IASSNS-HEP-88/39, to appear in Comm. Math. Phys.

[11] D. Gepner and E. Witten, Nucl. Phys. **B278** (1986) 493.

[12] E. Verlinde, Nucl. Phys. **300** (1988) 360.

[13] E. Verlinde, PhD thesis (1988).

[14] A.M. Polyakov and P.B. Wiegmann, Phys. Lett. **141B** (1984) 223.

[15] V.G. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. **B247** (1984) 83.

[16] T. Eguchi and H. Ooguri, Nucl. Phys. **B282** (1987) 308.

[17] D. Bernard, Nucl. Phys. **B303** (1988) 77; **B309** (1988) 145.

[18] M.F. Atiyah and R. Bott, Phil. Trans. R. Soc. Lond. A **308** (1982) 523.

[19] S.K. Donaldson, J. Diff. Geom. **18** (1983) 269; Duke Math. J. **54** (1987) 231.

[20] G. Felder, K. Gawedzki and A. Kupiainen, Nucl. Phys. **B299** (1988) 355; Comm. Math. Phys. **117** (1988) 127; K. Gawedzki, "Wess-Zumino-Witten Conformal Field Theory," IHES/P/89/06

[21] I. Frenkel, Invent. Math. **77** (1984) 301

[22] K. Gawedzki and A. Kupiainen, HU-TFT-88-29; 34; IHES/P/88/45.