String Theory I: Problem Sheet 2

Hilary Term 2019

[Last update: 09:14 on Thursday 31st January, 2019]

1. Charge algebras for the classical string

In Hamiltonian mechanics, conserved charges are represented by functions on phase space that Poisson-commute with the Hamiltonian, and the action of the symmetry corresponding to a charge on an observable is implemented by the Poisson bracket. In this exercise, you will calculate the algebra of the conserved charges associated to spacetime Poincaré symmetry and worldsheet conformal symmetry of the closed string, written in terms of oscillator coordinates on phase space, whose Poisson brackets take the canonical form

$$[\alpha_m^\mu, \alpha_n^\nu]_{\text{P.B.}} = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu]_{\text{P.B.}} = im\eta^{\mu\nu}\delta_{m+n,0} \ , \qquad [p^\mu, x^\nu]_{\text{P.B.}} = \eta^{\mu\nu} \ .$$

(*a*) Spacetime Poincaré symmetry charges for the closed string are written in terms of oscillator coordinates as follows:

$$\begin{aligned} P^\mu &= p^\mu \ , \\ M^{\mu\nu} &= x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^{\infty} \frac{\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu}{n} - i\sum_{n=1}^{\infty} \frac{\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu - \tilde{\alpha}_{-n}^\nu \tilde{\alpha}_n^\mu}{n} \ . \end{aligned}$$

Compute the Lie algebra of these charges.

(*b*) Derive the expression for the conserved charges L_m and $\tilde{L}_m$ used to impose the stress tensor constraints on the string phase space. Verify that their Lie algebra is the Witt algebra,

$$[L_m, L_n]_{\text{P.B.}} = i(m-n)L_{m+n} \ .$$

Show that the transformation of the oscillators under the action of these charges is given by

$$\begin{aligned} [L_m, \alpha_n^\mu]_{\text{P.B.}} &= -in\alpha_{m+n}^\mu \ . \\ [\tilde{L}_m, \tilde{\alpha}_n^\mu]_{\text{P.B.}} &= -in\tilde{\alpha}_{m+n}^\mu \ . \\ [L_m, \tilde{\alpha}_n^\mu]_{\text{P.B.}} &= [\tilde{L}_m, \alpha_n^\mu]_{\text{P.B.}} = 0 \ . \end{aligned}$$

Thus deduce the action of $\tilde{L}_m$ and L_m on the space-time coordinate fields $X^\mu(\tau,\sigma)$. Confirm that your results agree with the action of the vector fields generating worldsheet conformal transformations, $V_m^\pm = -\frac{1}{2}e^{2im\sigma^\pm}\partial_\pm$.[1]

2. Charge algebras for the quantum string

Upon quantization, Poisson brackets for the oscillator coordinates are promoted to commutation relations for corresponding creation and annihilation (and zero-mode) operators acting on the string Fock space,

$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0} \ , \qquad [p^\mu, x^\nu] = -i\eta^{\mu\nu} \ ,$$

where $\alpha_{-m}^\mu = (\alpha_m^\mu)^\dagger$ and similarly for the $\tilde{\alpha}$'s. In this exercise you will study the Fock space operators corresponding to the conserved charges encountered in our analysis of the classical string.

[1] Recall that the vector field $\xi^a(x)\partial_a$ generates a diffeomorphism $x^a \to \tilde{x}^a = x^a + \xi^a(x)$.

(a) Write the generators of spacetime Poincaré symmetry in terms of oscillator creation and annihilation operators. By direct computation or otherwise, show that the commutation relations for these charges are precisely those of the Poincaré algebra computed in problem 1(a).

(b) In the quantum theory, we define worldsheet conformal generators by

$$L_m = \tfrac{1}{2} \sum_{k=-\infty}^{\infty} \alpha_{m-k} \cdot \alpha_k \ ,$$

$$L_0 = \tfrac{1}{2}\alpha_0 \cdot \alpha_0 + \sum_{k=1}^{\infty} \alpha_{-k} \cdot \alpha_k \ .$$

Argue that the commutation relations of these operators must take the form

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0} \ ,$$

where $A(m)$ is a c-number (*i.e.*, just a number) depending on m. Now determine the form of $A(m)$, either by brute force, or by arguing as follows:

(i) First argue the $A(-m) = -A(m)$.

(ii) Now use the Jacobi identity for the commutator algebra to show that for $k+m+n=0$, one has

$$(n-m)A(k) + (k-n)A(m) + (m-k)A(n) = 0 \ .$$

(iii) Therefore deduce that in general the c-number term takes the form

$$A(m) = c_1 m + c_3 m^3 \ ,$$

where c_1 and c_3 are constant c-numbers.

(iv) By evaluating the expectation value of $[L_m, L_{-m}]$ in the oscillator ground state $|0;0\rangle$ for $m=1$ and $m=2$, determine the values of c_1 and c_3.

3. Open string at level *two*

Construct the reduced Hilbert space at level **two** for the open string with $a=1$. Give the D-dimensional spacetime interpretation of your results.

⋆ Bonus problem ⋆

In lecture we may or may not have time to perform in detail the crucial calculation of when additional physical spurious states exist at level two. If we don't do so, then perform the calculation now.

(i) Derive the conditions for a state of the form

$$|\psi\rangle = (L_{-2} + \gamma L_{-1}L_{-1})\,|0;p\rangle$$

to be spurious and physical.

(ii) Explain why this is the only form for an additional state (beyond those of the form $L_{-1}|\chi_1\rangle$) that one must examine at level two when looking for physical spurious states.

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