String Theory I: Problem Sheet 3

Hilary Term 2019

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1. Conformal properties of local operators

Recall that we called a boundary local operator $\mathcal{O}(\tau)$ a *primary operator of dimension* h if its commutation relations with the Virasoro charges take the form

$$[L_m, \mathcal{O}(\tau)] = e^{im\tau}\left(-i\frac{d}{d\tau} + mh\right)\mathcal{O}(\tau) \ .$$

On the second problem sheet you verified the closed-string version of the fact that the scalar fields $X^\mu(\tau)$ transform as primaries of dimension $(0,0)$, and in lecture it was stated that the boundary operator $X^\mu(\tau)$ transforms as a primary of dimension $h = 0$.

(a) Show that the first derivative of the boundary scalar field, $\partial_\tau X^\mu(\tau)$, is a primary field of dimension $h = 1$.

(b) Show that the *second derivative* of the boundary scalar field, $\partial_\tau^2 X^\mu(\tau)$, is not a primary field of any dimension.

(c) The vertex operator associated to absorption of an open string tachyon with spacetime momentum k is given as follows:

$$V(k;\tau) =: e^{ik\cdot X(\tau,0)}: \ = \exp\left(k\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}e^{in\tau}\right)e^{ik\cdot(x+p\,\tau)}\exp\left(-k\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{-in\tau}\right) \ .$$

Prove that this is a primary operator of dimension $h = k\cdot k/2$.

2. The dilaton in OCQ

We saw in lecture that at the first excited level of the closed string, there are $(D-2)^2$ physical states that organize into a graviton, an anti-symmetric two-form, and a scalar under space-time Poincaré transformations. I also wrote down an expression for the physical state corresponding to the scalar degree of freedom (the *dilaton*, usually denoted by ϕ).

(a) The first guess for the dilaton state in the closed-string Hilbert space was something of the form

$$|\phi;p\rangle \stackrel{?}{=} (\alpha_{-1}\cdot\tilde{\alpha}_{-1})\,|0;p\rangle \ ,$$

Confirm that this state does not satisfy the L_{+1} physical state condition. Find a way to improve this state to render it physical. Show that the resulting state that you write down represents a space-time scalar particle.

(b) Find a vertex operator $V_\phi(\tau,\sigma)$ for the dilaton. Check that it is a (bulk) primary operator of dimension $(1,1)$. Check that it satisfies the operator/state relation, [1]

$$\lim_{\tau\to i\infty} e^{-4i\tau}V_\phi(\tau,\sigma)|0;0\rangle = |\phi;p\rangle \ .$$

[1] Note that this is the operator/state relation relevant for the (slightly unusual!) GSW conventions for the closed string, where world-sheet time translations are generated by $H = 2(L_0 + \tilde{L}_0)$. It would look slightly more standard to define $\tau_{\text{new}} = 2\tau_{\text{old}}$, in terms of which we would have

$$\lim_{\tau\to i\infty} e^{-2i\tau_{\text{new}}}V_\phi(\tau_{\text{new}},\sigma)|0;0\rangle = |\phi;p\rangle \ .$$

3. String three-point couplings

In lecture, we will see/have seen that the three-point coupling of open string tachyons takes the extremely simple form (in conventions where all momenta are incoming)

$$\mathcal{A}_3(k_1, k_2, k_3) = g_o \langle 0; k_1 | V(k_2, \tau) | 0; k_3 \rangle = g_o\, \delta(k_1 + k_2 + k_3) \ ,$$

where g_o is the open string coupling constant.

(a) Compute the three-point coupling of two open string tachyons with one photon.

(b) Compute the three-point coupling of one open string tachyon with two photons.

(c) Verify that these couplings respect gauge invariance of the photon.

4. Products of vertex operators and the Veneziano amplitude

In the context of string scattering amplitudes for four or more particles, one necessarily has to deal with products of vertex operators. This made easier by being able to normal order the products.

(a) Prove the formula

$$V(k_1, \tau_1) V(k_2, \tau_2) =: V(k_1, \tau_1) V(k_2, \tau_2) : \left(e^{i\tau_1} - e^{i\tau_2}\right)^{k_1 \cdot k_2}$$

where the colons around the product mean that the entire product is defined according to creation/annihilation normal ordering conventions.

This leaves some ambiguity as to the ordering of the zero modes x^μ and p^μ, and you should find the appropriate ordering rule so that the result takes the given simple form. If necessary, give τ_1 and τ_2 small imaginary parts and make whatever assumptions are necessary about their relative magnitudes.

(b) Use your result from part (a) to derive an integral expression for the Veneziano amplitude for the scattering of four open string tachyons using the general form for the four-point amplitude

$$\mathcal{A}_4(k_1, k_2, k_3, k_4) = g_o^2 \int_{-\infty}^{0} d\mathbf{t}\ \langle 0; k_1 | V(k_2, 0) V(k_3, \tau = -i\mathbf{t}) | 0; k_4 \rangle \ .$$

By looking up "Euler beta function" in a reference of your choice, confirm that your result does indeed match the Veneziano amplitude we studied in Problem Sheet 1. Comment on the conditions for convergence of the integral.

(c) [**Bonus**] Derive an integral expression for n-point generalization of the Veneziano amplitude (*i.e.*, the n-point open-string tachyon scattering amplitude). Don't worry about evaluating the integral!

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