Gaussian Fields and Percolation

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RANDOM WAVES IN OXFORD

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Berry's conjecture

In 1977 M. Berry conjectured that high energy eigenfunctions in the chaotic case have statistically the same behaviour as random plane waves. (Figures from Bogomolny-Schmit paper)

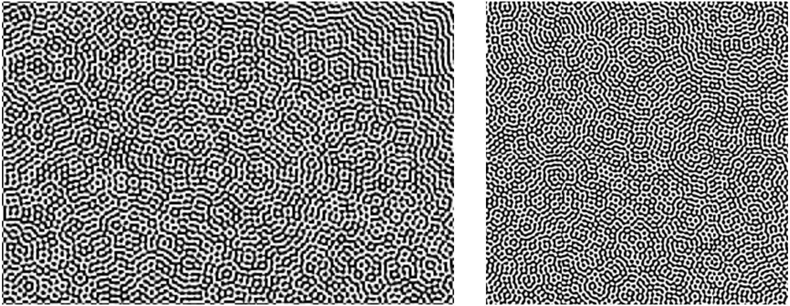


Figure: Nodal domains of an eigenfunction (left) of a stadium and of a random plane wave (right)

Random Plane Wave

Two ways to (informally) think of the random plane wave

- A "random" or "typical" solution of Helmholtz equation $\Delta f + k^2 f = 0$
- A random superposition of all possible plane waves with the same frequency k

The second approach leads to a naive definition that it is the limit of

$$\Psi_n(z) = \mathrm{Re}\left(\sum_{j=1}^{n} e^{k(\theta_j, z) + \phi_j}\right)$$

where θ_j are uniform random directions and ϕ_j are random phases.

Gaussian functions and fields

Two ways to define Gaussian random functions

- Random series ϕ_i orthonormal basis in some Hilbert space H

$$\Psi = \sum a_i \phi_i, \qquad a_i \text{ i.i.d. } N(0,1)$$

- Gaussian field $\Psi(x)$ is a collection of jointly Gaussian random variables indexed by x. Could be defined by its covariance function $K(x,y) = \mathbb{E}\left[\Psi(x)\Psi(y)\right]$. Mostly interested in stationary case $K(x,y) = K(x-y)$.

Covariance function

$$K(x,y) = \sum \phi_i(x)\phi_i(y)$$

Stationary Gaussian functions

- Hilbert space H with a reproducing kernel $K(x, y)$. Take any orthonormal basis ϕ_i and construct $f = \sum a_i \phi_i$. The result is not in H but independent of the basis.
- This is a Gaussian field with covariance kernel $K(x, y)$.
- If $K(x, y) = K(x - y)$ then K is a positive definite function and its Fourier transform is a positive measure ρ. It is called the spectral measure.

Properties of f, H, K, and ρ are closely related. In particular, smoothness of K at zero or finite moments of ρ imply smoothness of f.

Random Plane Wave

Consider $L^2_s(\mathbb{T})$ – the Hilbert space of L^2 functions on the unit circle that satisfy symmetry condition $\phi(-z) = \overline{\phi(z)}$. We define H to be inverse 2d Fourier transform of L^2_s with scalar product inherited from L^2. This space consist of real analytic functions satisfying Helmholtz equation. Standard basis in $L^2(\mathbb{T})$ is $e^{in\theta}$. This leads to

$$f(z) = f(re^{i\theta}) = \sum C_n J_{|n|}(r) e^{in\theta}$$

where $C_n = \overline{C}_{-n}$ are independent Gaussian random variables and J_n are Bessel functions.
The covariant kernel is $J_0(|z|)$ and the spectral measure is $d\theta/2\pi$.

Related Fields: Random Spherical Harmonic

Consider H_n the space of all spherical harmonic of degree n with L^2 norm. This is $2n+1$ dimensional space. A Gaussian vector g_n in this space is the random spherical harmonic.

Note: H_n is an eigenspace of spherical Laplacian with eigenvalue $n(n+1)$.

Covariance kernel

$$\mathbb{E}\left[g(x)g(y)\right] = P_n(\cos(\theta(x,y)))$$

where P_n is the Legendre polynomial of degree n normalized by $P_n(1) = 1$ and $\theta(x,y)$ is the angle between x and y (i.e. spherical distance).

Scaling Limit of Random Spherical Harmonics

Theorem (Zelditch)

Random plane wave is the scaling limit of random spherical harmonic

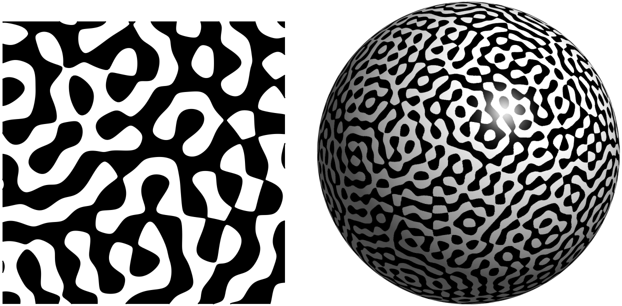


Figure: Nodal lines of a random plane wave and of a random spherical harmonic

Universality of Random Plane Waves

Let $(\mathcal{M}, g)$ be a compact Riemannian manifold, ϕ_i o.n.b. in $L^2(\mathcal{M})$ of eigenfunctions

$$\Delta\phi_i + \lambda_i^2\phi_i = 0, \qquad \lambda_i \leq \lambda_{i+1}$$

Band-limited function

$$f_n(x) = \sum_{n^2-n\leq i\leq n^2} c_i\phi_i(x)$$

Scaling limit on the tangent plane: for $x_0 \in \mathcal{M}$ define

$$F_n(x) = f_n(\exp_{x_0}(x/n))$$

where $\exp_{x_0} : T_{x_0}\mathcal{M} \to \mathcal{M}$ is the exponential map. Then F_n converges to the random plane wave as $n \to \infty$.

Deterministic Results

Some universal estimates are known for eigenfunctions of Laplacian.

Theorem

Nodal set for random plane wave forms a c/λ-net where c is an absolute constant. Nodal set for spherical harmonic forms a c/n-net.

Theorem

Every nodal component contains a disc of radius c/λ (or c/n) where c is an absolute constant.

Length of Nodal Lines

Theorem

There is a constant c such that for every spherical harmonic g_n of degree n such that

$$\frac{n}{c} < L(g_n) < cn$$

where $L(g_n)$ is the length of nodal set.

Yau conjecture: For a compact C^∞ smooth Riemannian manifold $\mathcal{M}$ there is $c > 0$ such that for every eigenfunction $\Delta\phi + \lambda^2\phi = 0$

$$\lambda/c \leq H^{n-1}(\phi = 0) \leq c\lambda$$

In dimension $n = 2$ lower bound by Brüning (1978). For $n > 2$ in real-analytic case by Donnelly-Fefferman (1988), the lower bound in C^∞ case by Logunov (2016).

Nodal Lines of Gaussian Spherical Harmonic

Theorem (Bérard, 1985)

For Gaussian spherical harmonic g_n of degree n

$$\mathbb{E}L(g_n) = \pi\sqrt{2}\lambda_n = \sqrt{2}\pi n + O(1)$$

With more careful analysis of Kac-Rice formula it is possible to compute variance

Theorem (Wigman, 2009)

For Gaussian spherical harmonic g_n of degree n

$$\operatorname{Var} L(g_n) = \frac{1}{32}\ln(n) + O(1)$$

Number of Nodal Domains

In the deterministic case Courant's theorem gives that the number of nodal domains $N(g_n) < n^2$. In 1956 Pleijel improved the upper bound to $0.69n^2$. For $n > 2$ Lewy constructed spherical harmonic with two or three nodal domains, so there is no non-trivial deterministic lower bound.

The main problem: this is a non-local quantity.

Theorem (Nazarov and Sodin, 2007)

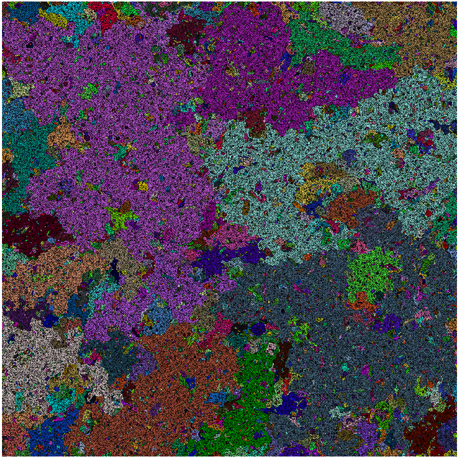
Let g_n be Gaussian spherical harmonic of degree n. Then there is a positive constant a such that

$$\mathbb{P}\left\{\left|\frac{N(g_n)}{n^2} - a\right| > \epsilon\right\} \leq C(\epsilon)e^{-c(\epsilon)n}$$

where $C(\epsilon)$ and $c(\epsilon)$ are positive constant depending on ϵ only.

Nodal Domains

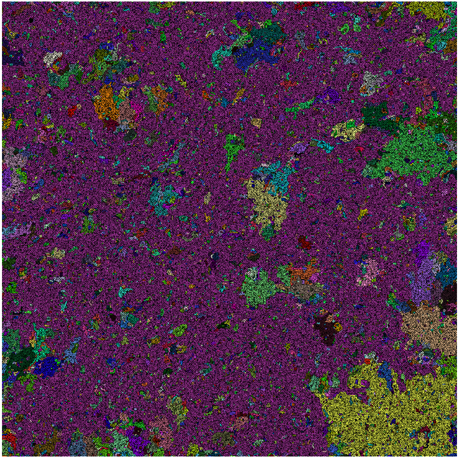
All positive nodal domains of a random plane wave.



Picture by T. Sharpe.

Nodal Domains

All negative nodal domains of a random plane wave.



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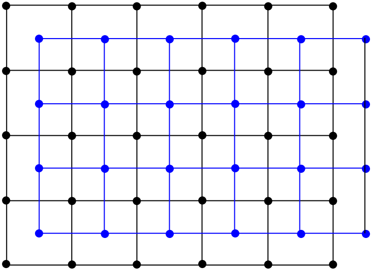
Critical Square Lattice Bond Percolation

Each edge of the lattice is preserved with probability $p_c = 1/2$. If an edge is preserved, then the dual edge is removed and vice versa. Primal and dual clusters create an loop model of interfaces.



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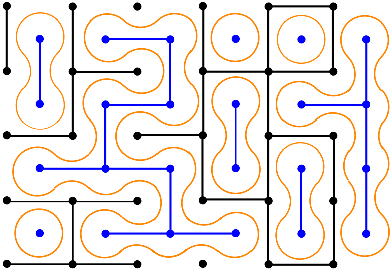

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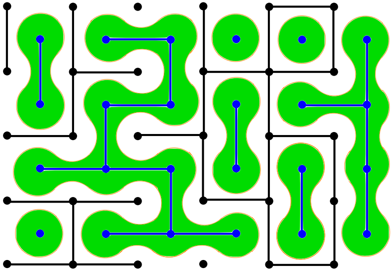
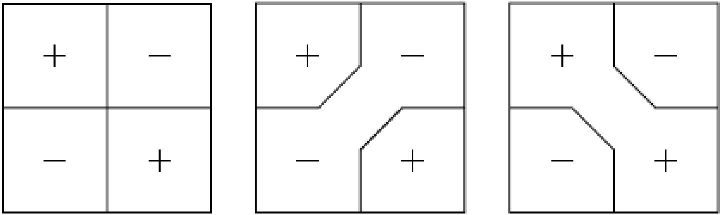


Bogomolny-Schmit Percolation Model

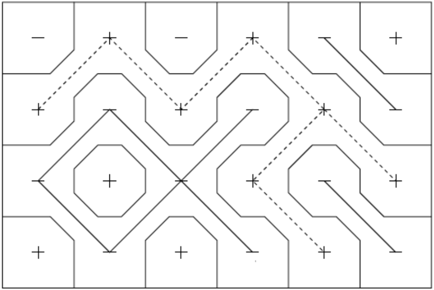
They proposed think that the nodal lines form a perturbed square lattice



Picture from Bogomolny-Schmit paper.

Bogomolny-Schmit Percolation Model

Using this analogy we can think of the nodal domains as percolation clusters on the square lattice.



This leads to the conjecture that

$$\mathbb{E}(N(f), \Omega) = \mathrm{Area}(\Omega)\frac{3\sqrt{3}-5}{4\pi^2}$$

Off-critical Percolation

Off-critical percolation is a model for excursion and level sets

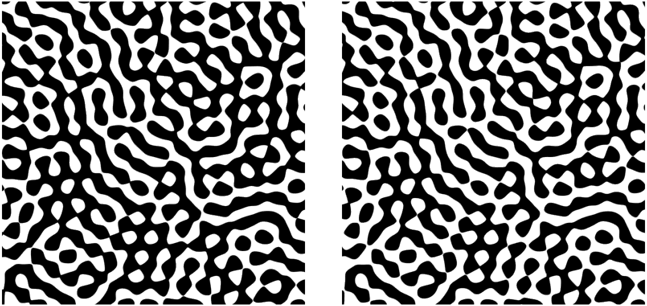


Figure: Excursion sets for levels 0 (nodal domains) and level 0.1

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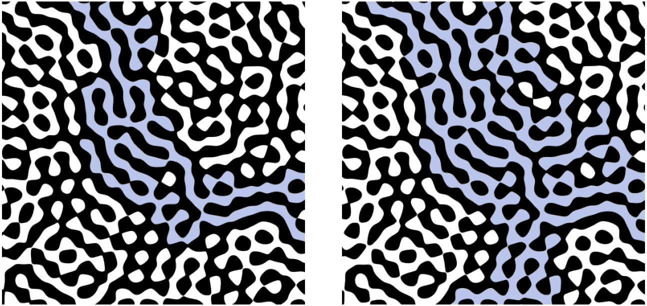


Figure: Excursion sets for levels 0 (nodal domains) and level 0.1

Is It Really True?

- Numerical results (Nastasescu (2011), Konrad (2012), B.-Kereta (2013)) show that the number of nodal domains per unit area is 0.0589 instead of 0.0624 predicted by Bogomolny-Schmit.
- Number of clusters per vertex is a non-universal quantity in percolation, it is lattice dependent. Global properties should be universal i.e. lattice independent.
- Numerical evidence that many global 'universal' observables (crossing probabilities, decay rate for the area of nodal domains, one-arm exponent) match percolation predictions.

Universality Class

This seems to be a rather universal phenomenon. For a wide class of smooth stationary fields their nodal domains are in the same universality class as critical percolation.
Assumptions:

- Smooth (nodal lines are nice curves)
- Stationary (percolation is almost stationary)
- Isotropic or symmetric enough (uniform conformal structure)
- Weakly correlated (percolation is local)

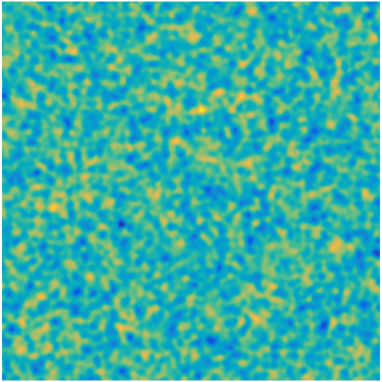
A Good Example

Bargmann-Fock function

$$f(x) = \sum a_{i,j} \frac{1}{\sqrt{i!j!}} x_1^i x_2^j e^{-|x|^2/2}$$

Covariance kernel

$$K(x,y) = e^{-|x-y|^2/2}$$



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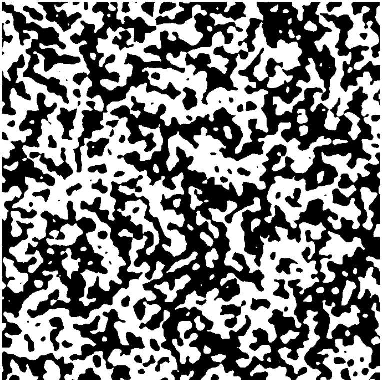

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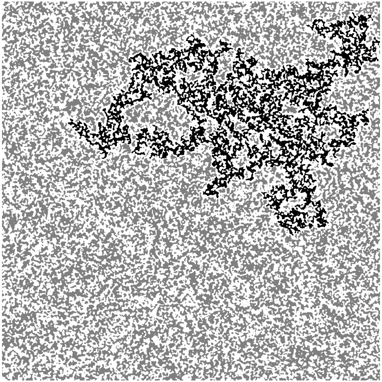
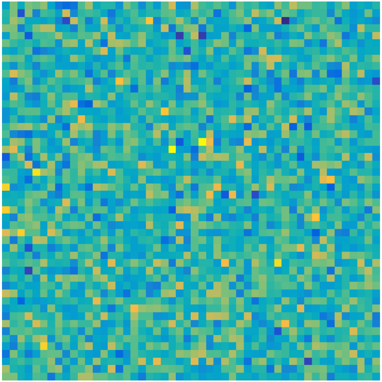


A Bad Example

White noise on the square lattice

Nodal domains are exactly Bernoulli site percolation clusters with $p = 1/2$ which is not critical.



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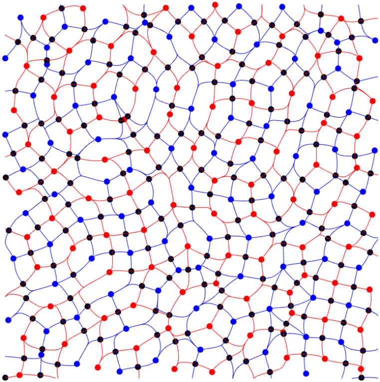
An Ugly Example

Gradient flow percolation model.

Nodal domains could be modelled by a lattice model. Not clear how to analyse.



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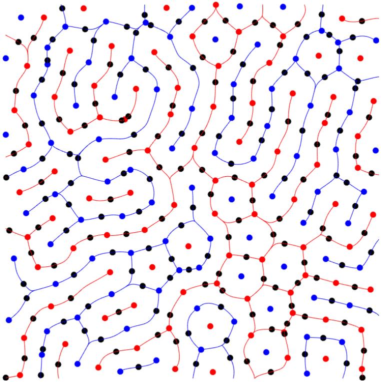

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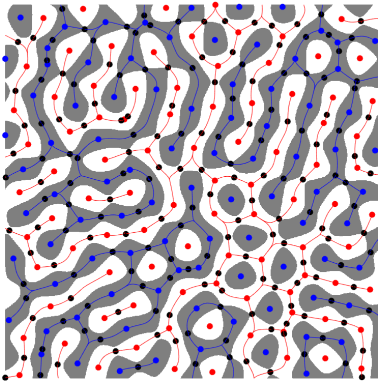


What Do We Know

- Molchanov-Stepanov 1983: For sufficiently regular fields excursion sets percolate at high levels
- Alexander 1996: For ergodic positively correlated fields level lines never percolate
- Many local (lengths, areas) and semi-local (number of excursion sets) quantities have scaling limits under very mild regularity assumptions. (Kac, Rice, Berard, Nazarov, Sodin and many others)

We expect that only global observables have universal behaviour.

Conformally Invariant Scaling Limits

General strategy:

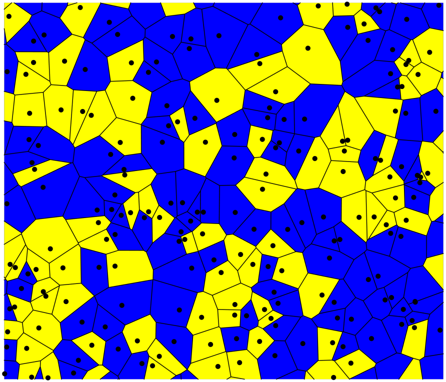
1. Show tightness/pre-compactness which would imply existence of subsequential limits.
2. Show that one global observable has conformally invariant scaling limit
3. Show that above implies that the curves are described by Loewner Evolution and all subsequential limits are driven by Brownian motion

Global Observable: Crossing Probability

- Crossing probability is one of the fundamental observables.
- Smirnov: Existence of conformally invariant scaling limit of crossing probabilities implies convergence of interfaces to SLE. Cardy's formula is very hard to prove.
- Russo-Seymour-Welsh estimates: bounds on crossing probability that are independent of scale. Hard to work with non-local models

Tassion: RSW for Voronoi Percolation

Tassion proved RSW for percolation on Voronoi tessellation generated by a Poisson point process



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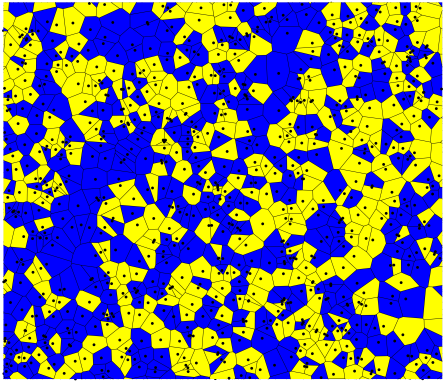
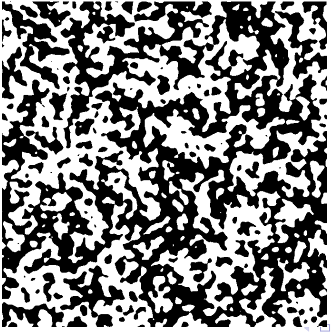


Russo-Seymour-Welsh

Theorem (Beffara-Gayet 2016)

Russo-Seymour-Welsh estimate for Bargmann-Fock random function.

Important: Covariance is positive, symmetric, fast decaying.
Decay rate could be improved: B.-Muirhead, Rivera-Vanneuville.



Russo-Seymour-Welsh

Theorem (B.-Muirhead-Wigman)

Russo-Seymour-Welsh estimate for Kostlan ensemble

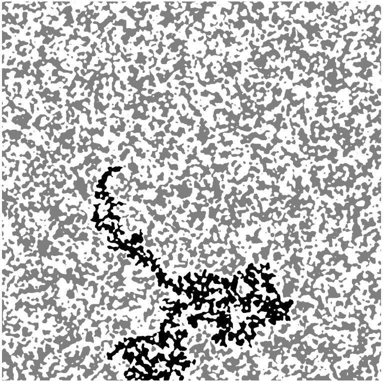
Kostlan or complex Fubini-Study ensemble of homogeneous polynomials $\mathbb{R}^3$ (or S^2)

$$f(x) = \sum_{|J|=n} a_J \sqrt{\binom{n}{J}} x^J$$

Covariance kernel $\cos^n(d(x,y))$.
Locally converges to Bargmann-Fock.

Off-critical crossing

B., Muirhead, Rivera, Vanneuville: For a wide class of symmetric positively correlated fields level sets exhibit sharp transition at 0 similar to sharp transition in percolation



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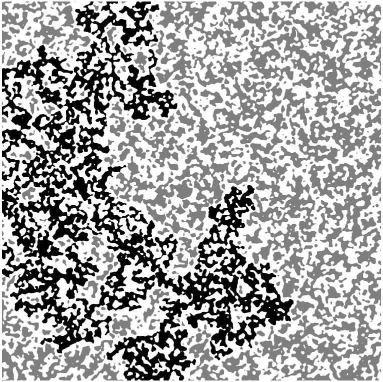

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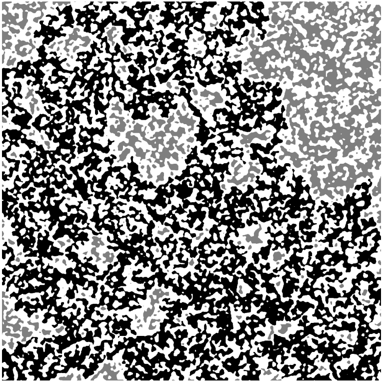

Thank you!