Anisotropic Random Wave Models

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What is the talk about?

k is a random vector in $\mathbb{R}^d$ $\left(d \geq 2\ ,\ \mathbf{k} \neq 0\right)$

Associate

- covariance function: $t \in \mathbb{R}^d \mapsto \mathbb{E}\cos(\mathbf{k} \cdot t)$
- Gaussian random field on $\mathbb{R}^d$, say $G_{\mathbf{k}}$, that is stationary centered with such a covariance

Question:

- links between anisotropy properties of $G_{\mathbf{k}}$ and those of **k** ?

Outline of the talk

1. Random wavevector and associated covariance function
2. Level sets of Gaussian waves
3. Crest lines in the planar case

without isotropy hypothesis

1. Random wavevector

$\mathbf{k}$ is a random vector in $\mathbb{R}^d$ such that $\mathbb{P}(\mathbf{k} = 0) = 0$ (wavevector)

Notations

- matrix $\mathbf{k}\mathbf{k}^T = (\mathbf{k}_i\mathbf{k}_j)_{1 \leq i,j \leq d}$
- $\mathbf{k} = R\widetilde{\mathbf{k}}$ with $R = \|\mathbf{k}\|$ and $\widetilde{\mathbf{k}} \in \mathbb{S}^{d-1}$
- $d\mu(\lambda)$: probability distribution of $\mathbf{k}$ on $\mathbb{R}^d$

Vocabulary

- $\mathbf{k}$ is isotropic if $\widetilde{\mathbf{k}}$ is uniformly distributed in $\mathbb{S}^{d-1}$
- $\mathbf{k}$ is separable if $\|\mathbf{k}\|$ and $\widetilde{\mathbf{k}}$ are independent random variables

Particular cases

- $\|\mathbf{k}\| = \kappa$, *a.s.* with κ constant > 0 (wavenumber)
  note that **k** is separable in that case

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- $d = 2$, $\mathbf{k}$ separable, $\widetilde{\mathbf{k}} = (\cos\Theta, \sin\Theta)$ with
  - $\Theta \sim \mathcal{U}([0, 2\pi])$ (isotropic case)
  - or $\Theta \sim \mathcal{U}([-\delta, \delta])$ (elementary case)
  - or $\Theta \sim C_\alpha |\cos\theta|^\alpha \, d\theta$ (toy model)

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  - or $\Theta \sim C_\alpha |\cos\theta|^\alpha \, d\theta$ (toy model)
- $d = 3$ and $\mathbf{k} \in \mathcal{A} = \{x^2 + y^2 = z^4\}$ *a.s.* (Airy surface)

Rmk: In examples 1 and 3, $\mathbf{k}$ is such that $Pol(\mathbf{k}) = 0$

Single random wave

Let $\mathbf{k}$ be a random wavevector in $\mathbb{R}^d$
Let η be a r.v. independent of $\mathbf{k}$ with $\eta \sim \mathcal{U}([0, 2\pi])$ and

$$X(t) = \sqrt{2}\cos(\mathbf{k} \cdot t + \eta), \ t \in \mathbb{R}^d$$

Hence

- X is centered, variance 1
- X is second order stationary with $\mathbb{E}[X(s)X(t)] = \mathbb{E}\cos(\mathbf{k} \cdot (t - s))$
- X is not second order isotropic (unless $\mathbf{k}$ is isotropic)

Gaussian random wave associated with a wavevector

Let $\mathbf{k}$ be a random wavevector in $\mathbb{R}^d$

Def: We call Gaussian random wave associated with $\mathbf{k}$ any Gaussian random field G on $\mathbb{R}^d$ that is stationary and centered with covariance

$$r(t) := \mathbb{E}(G(t)G(0)) = \mathbb{E}\cos(\mathbf{k}\cdot t),\ t \in \mathbb{R}^d$$

Rmk: $\mathrm{Var}\, G(0) = 1$ and

$$r(t) = \int_{\mathbb{R}^d} e^{i\lambda\cdot t} d\mu^{(s)}(\lambda)$$

with $\mu^{(s)} = \frac{1}{2}(\mu + \check{\mu})$ the spectral measure of G

Covariance function

$\mathbf{k}$ is a random vector in $\mathbb{R}^d$ and $r(t) = \mathbb{E}\cos(\mathbf{k}\cdot t)$, $t \in \mathbb{R}^d$

Fact:

- r is of class $\mathcal{C}^m$ iff $\mathbf{k}$ admits finite moments of order m
- for any $\mathbf{j} = (j_1, \dots, j_d)$, $\partial^{\mathbf{j}} r(0) = 0$ if $|\mathbf{j}|$ is odd and

$$\partial^{\mathbf{j}} r(0) = (-1)^{|\mathbf{j}|/2}\mathbb{E}\mathbf{k}^{\mathbf{j}} \quad \text{if} \quad |\mathbf{j}| \quad \text{is even}$$

- $\mathbb{E}(G'(0)G'(0)^T) = -r''(0) = \mathbb{E}(\mathbf{k}\mathbf{k}^T)$ ($d \times d$ matrix)

Partial Differential Equation

P multivariate even polynomial: $P(\lambda) = \sum_{\mathbf{j} \in \mathbb{N}^d;\, |\mathbf{j}| \text{ even}} \alpha_{\mathbf{j}} \, \lambda^{\mathbf{j}}$

$\mathcal{L}_P = \sum_{\mathbf{j} \in \mathbb{N}^d;\, |\mathbf{j}| \text{ even}} (-1)^{|\mathbf{j}|/2} \alpha_{\mathbf{j}} \, \partial^{\mathbf{j}}$: differential operator

Let $\mathbf{k}$ be a wavevector in $\mathbb{R}^d$ and G associated Gaussian wave

G is an a.s. solution of $\mathcal{L}_P(G) = 0$

$\Leftrightarrow \; P(\mathbf{k}) = 0 \, a.s.$

$\Leftrightarrow$ spectral measure of G supported by $\{\lambda \in \mathbb{R}^d \; : \; P(\lambda) = 0\}$

Examples

- Berry random wave: $\|\mathbf{k}\| = \kappa$ *a.s.* with κ *constant* > 0
  Gaussian wave G satisfies $\Delta G + \kappa^2 G = 0$ *a.s.*

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- Sea waves: $\mathbf{k}$ in $\mathbb{R}^3$ with $(k_x)^2 + (k_y)^2 = (k_t)^4$, *a.s.*
  Gaussian wave G on $\mathbb{R}^2 \times \mathbb{R}$: height at point (x, y) at time t. It satisfies $\Delta G + \frac{\partial^4}{\partial t^4} G = 0$ *a.s.*

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- Acoustic/optical waves in heterogeneous media, ...

2. Level sets

Let $\mathbf{k}$ random wavevector in $\mathbb{R}^d$,
G associated Gaussian random field defined on $\mathbb{R}^d$,
$a \in \mathbb{R}$ fixed level

$$G^{-1}(a) = \{t \in \mathbb{R}^d : \; G(t) = a\},$$

- submanifold of $\mathbb{R}^d$, dimension $d-1$
- nodal set in the case $a = 0$
- $\forall t \in G_{\mathbf{k}}^{-1}(a)$, tangent space $T_t G_{\mathbf{k}}^{-1}(a)$ is $\perp$ $G'(t)$

question: "favorite" orientation of $T_t G^{-1}(a)$?

Favorite orientation of level sets

<u>def:</u> favorite direction of V (V: rdom in $\mathbb{R}^d$) is any direction in

$$\text{Argmax}\,\{\mathbb{E}(V.u)^2\,;\ u \in \mathbb{S}^{d-1}\}$$

But $\mathbb{E}(V.u)^2 = u.\mathbb{E}(VV^T)u$ and $\mathbb{E}(G'(0)G'(0)^T) = \mathbb{E}(\mathbf{k}\mathbf{k}^T)$
so, morally: "The favorite orientation(s) of the level sets $G^{-1}(a)$ is(are) orthogonal to the favorite direction(s) of $\mathbf{k}$"

"It becomes highly probable that the direction of the contour is near the principal direction" [Longuet-Higgins'57]

$(d = 2)$ Favorite direction of level lines - examples

Let $\mathbf{k}$ separable, so $\mathbb{E}(\mathbf{k}\mathbf{k}^T) = (\mathbb{E}\|\mathbf{k}\|^2)\mathbb{E}(\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T)$
and let $\widetilde{\mathbf{k}} = (\cos\Theta, \sin\Theta)$

- isotropic case: $\Theta \sim \mathcal{U}([0, 2\pi])$
  $\mathbb{E}(\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T) = I_2$ then, no favorite direction
- toy model: $\Theta \sim C_\alpha \,|\cos\theta|^\alpha \, d\theta$ with some $\alpha > 0$
  $\mathbb{E}(\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T) = \frac{1}{\alpha+2}\begin{pmatrix} \alpha+1 & 0 \\ 0 & 1 \end{pmatrix}$
  favorite direction of level lines is $\perp 0$
- elementary model $\Theta \sim \mathcal{U}([-\delta, \delta])$ with some $\delta \in (0, \pi/2)$
  $\mathbb{E}(\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T) = \begin{pmatrix} 1+\mathrm{sinc}(2\delta) & 0 \\ 0 & 1-\mathrm{sinc}(2\delta) \end{pmatrix}$
  favorite direction of level lines is $\perp 0$

Expected measure of level sets

Let Q compact $\subset \mathbb{R}^d$. Kac-Rice formula yields

$$\mathbb{E}[\mathcal{H}_{d-1}(G^{-1}(a) \cap Q)] = \int_Q \mathbb{E}[\| G'_{\mathbf{k}}(t) \| \mid G_{\mathbf{k}}(t) = a] \, p_{G_{\mathbf{k}}(t)}(a) \, dt$$

$$= \mathcal{H}_d(Q) \, \frac{e^{-a^2/2}}{\sqrt{2\pi}} \, \mathbb{E}\| G'_{\mathbf{k}}(0) \|$$

$$\text{with} \quad \mathbb{E}\| G'_{\mathbf{k}}(0) \| = \int_{\mathbb{R}^d} (\mathbb{E}(\mathbf{k}\mathbf{k}^T) x \cdot x)^{1/2} \Phi_d(x) \, dx$$

Separable case: $\mathbf{k} = \|\mathbf{k}\| \widetilde{\mathbf{k}}$ with $\|\mathbf{k}\| \perp\!\!\!\perp \widetilde{\mathbf{k}}$, then

$$\mathbb{E}\| G'_{\mathbf{k}}(0) \| = (\mathbb{E}\|\mathbf{k}\|^2)^{1/2} \int_{\mathbb{R}^d} (\mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T] x \cdot x)^{1/2} \Phi_d(x) \, dx$$

Expected measure of level sets - Berry isotropic RW

- Berry isotropic case: $\|\mathbf{k}\| = \kappa$ and $\widetilde{\mathbf{k}} \sim \mathcal{U}(\mathbb{S}^{d-1})$

$$\mathbb{E}[\mathcal{H}_{d-1}(G^{-1}(a) \cap Q)] = \mathcal{H}_d(Q) \, \frac{e^{-a^2/2}}{\sqrt{2\pi}} \, \kappa \, \frac{\Gamma((d+1)/2)}{\Gamma(d/2)}$$

- Berry isotropic planar case, nodal line $(d = 2, a = 0)$

$$\mathbb{E}[length(G^{-1}(a) \cap Q)] = \mathcal{H}_2(Q) \, \frac{1}{2\sqrt{2}} \, \kappa$$

Planar case - Mean length of level curves

$$\mathbb{E}[length(G^{-1}(a) \cap Q)] = \mathcal{H}_2(Q)\, \frac{e^{-a^2/2}}{\sqrt{2\pi}}\, \mathbb{E}\|G'_{\mathbf{k}}(0)\|$$

with

$$\begin{aligned} E\|G'_{\mathbf{k}}(0)\| &= \int_{\mathbb{R}^2} (\mathbb{E}(\mathbf{k}\mathbf{k}^T)x \cdot x)^{1/2}\, \Phi_2(x)\, dx \\ &= (2/\pi)^{1/2}\, (\gamma_+)^{1/2}\, \mathcal{E}\big((1-\gamma_-/\gamma_+)^{1/2}\big), \end{aligned}$$

where

- $\mathcal{E}(x) = \int_0^{\pi/2} (1 - x^2 \sin^2\theta)^{1/2} d\theta$, elliptic integral
- $0 \le \gamma_- \le \gamma_+$ are the eigenvalues of $\mathbb{E}(\mathbf{k}\mathbf{k}^T)$

Mean length of level curves - separable case

separable case: $\mathbf{k} = \|\mathbf{k}\|\widetilde{\mathbf{k}}$ with $\|\mathbf{k}\| \perp\!\!\!\perp \widetilde{\mathbf{k}}$ then

- $\mathbb{E}(\mathbf{k}\mathbf{k}^T) = (\mathbb{E}\|\mathbf{k}\|^2)\ \mathbb{E}(\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T)$
- $\gamma_\pm = (\mathbb{E}\|\mathbf{k}\|^2)\,\widetilde{\gamma}_\pm$ and $\widetilde{\gamma}_+ + \widetilde{\gamma}_- = Trace(\mathbb{E}(\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T)) = 1$

hence

$$\mathbb{E}[length(G^{-1}(a) \cap Q)] = \mathcal{H}_2(Q)\,\frac{e^{-a^2/2}}{\pi\sqrt{2}}\,(\mathbb{E}\|\mathbf{k}\|^2)^{1/2}\,\mathcal{F}(c(\widetilde{\mathbf{k}}))$$

where the map $\mathcal{F} : c \in [0,1] \mapsto (1+c)^{1/2}\,\mathcal{E}\left(\left(\frac{2c}{1+c}\right)^{1/2}\right)$

is strictly decreasing

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- but what about $c(\widetilde{\mathbf{k}})$?

Coherency index

Def: the coherency index of matrix M is: $\dfrac{\gamma_+ - \gamma_-}{\gamma_+ + \gamma_-}$

where $0 \leq \gamma_- \leq \gamma_+$ are the eigenvalues of M

$$c(\mathbf{k}) = \text{ the coherency index of } \mathbb{E}(\mathbf{k}\mathbf{k}^T).$$

Result: if $\mathbf{k}$ is separable,

- $c(\mathbf{k}) = c(\widetilde{\mathbf{k}})$ only depends on the directional distrib. of $\mathbf{k}$
- and

$$\mathbb{E}[length(G^{-1}(a) \cap Q)] \text{ is a } \searrow \text{ function of } c(\widetilde{\mathbf{k}})$$

Coherency index as anisotropy parameter (examples)

separable case: $\mathbf{k} = \|\mathbf{k}\| (\cos\Theta, \sin\Theta)$ with $\|\mathbf{k}\| \perp\!\!\!\perp \Theta$

- Toy model: $\Theta \sim C_\alpha |\cos\theta|^\alpha \, d\theta$

$$c(\widetilde{\mathbf{k}}) = \alpha \ (\nearrow \text{ function of } \alpha)$$

- Elementary model: $\Theta \sim \mathcal{U}([-\delta, \delta] \cup [\pi - \delta, \pi + \delta])$

$$c(\widetilde{\mathbf{k}}) = \text{sinc}(2\delta) \ (\searrow \text{ function of } \delta \in [0, \pi/2])$$

3. Crest lines

k a 2-dim rdom wavevector, G associated Gaussian wave
$\varphi \in [0, \pi)$ fixed, $u_\varphi = (\cos\varphi, \sin\varphi)$

$$Z_\varphi := G' \cdot u_\varphi = \{G'(t) \cdot u_\varphi \,;\, t \in \mathbb{R}^2\}$$

$$Z_\varphi^{-1}(0) = \text{nodal line of } Z_\varphi := \text{crest line in direction } \varphi$$

<u>Claim:</u> Z_φ Gaussian wave associated with rdom wavevector $\mathbf{K}_\varphi$

$$\mathbf{K}_\varphi \sim (\lambda \cdot u_\varphi)^2 \, \frac{d\mu(\lambda)}{m_{20}(\varphi)}$$

with

$$m_{ij}(\varphi) = \int (\lambda \cdot u_\varphi)^i (\lambda \cdot u_{\varphi+\pi/2})^j \, d\mu(\lambda) = \int (\lambda_1)^i (\lambda_2)^j \, d\mu_\varphi(\lambda)$$

Mean length of crest lines

$$\mathbb{E}[length(Z_{\varphi}^{-1}(0) \cap Q)] = \mathcal{H}_2(Q)\,\frac{1}{\sqrt{2\pi}\, m_{20}(\varphi)}\,\mathbb{E}\|Z'_{\varphi}(0)\|$$

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- are equal to the eigenvalues of $\mathbb{E}[R_{-\varphi}(\mathbf{K}_\varphi)R_{-\varphi}(\mathbf{K}_\varphi)^T]$
  $\Longrightarrow$ 2 distinct formulas !

Mean length of crest lines - separable case

k separable: $\mathbf{k} = \|\mathbf{k}\|\widetilde{\mathbf{k}}$ with $\|\mathbf{k}\| \perp\!\!\!\perp \widetilde{\mathbf{k}}$.
It implies

- $\mathbf{K}_\varphi$ is separable, $\mathbf{K}_\varphi = \|\mathbf{K}_\varphi\|\,\widetilde{\mathbf{K}_\varphi}$
- $\mathbb{E}[\|\mathbf{K}_\varphi\|^2] = M_4/M_2$: indep of φ, with $M_j = \mathbb{E}\|\mathbf{k}\|^j$
- $c(\mathbf{K}_\varphi) = c(\widetilde{\mathbf{K}}_\varphi)$: depends on φ and on (4th moment of) $\widetilde{\mathbf{k}}$

hence

$$\mathbb{E}[length(Z_\varphi^{-1}(0) \cap Q)] = \mathcal{H}_2(Q)\,(M_4/M_2)^{1/2}\,\mathcal{F}(c(\widetilde{\mathbf{K}_\varphi}))$$

where the map $\mathcal{F}$ is strictly decreasing

In which direction is the longuest crest ?

- Rule of thumb: *"the direction that maximises the expected length of crests is orthogonal to the direction for the maximum integral of the spectrum, i.e. the most probable direction for the waves"*

In which direction is the longest crest ?

- Rule of thumb: *"the direction that maximises the expected length of crests is orthogonal to the direction for the maximum integral of the spectrum, i.e. the most probable direction for the waves"*
- Computational answer: $Argmax_{\varphi}\, c(\widetilde{\mathbf{K}_{\varphi}})$

Recall we have 2 formulas, but none is tractable ... until now!

Longuest crest - examples

Question: $Argmax_{\varphi}\, c(\widetilde{\mathbf{K}_{\varphi}}) = ?$

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Question: $Argmax_{\varphi}\, c(\widetilde{\mathbf{K}_{\varphi}}) = ?$

- $\widetilde{\mathbf{k}}$ isotropic
  $\Rightarrow c(\widetilde{\mathbf{K}_{\varphi}}) = 0$, there is no maximum
- $\widetilde{\mathbf{k}} \sim \frac{1}{4}(\delta_0 + \delta_{\pi/2} + \delta_{\pi} + \delta_{3\pi/2})$
  $\Rightarrow c(\widetilde{\mathbf{K}_{\varphi}}) = |\cos 2\varphi|$, max for $\varphi = \pi/4$ or $3\pi/4$

Longuest crest - elementary case

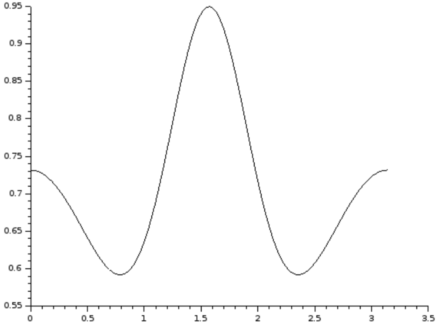
Let $\widetilde{\mathbf{k}} \sim \mathcal{U}([-\delta, \delta] \cup [\pi - \delta, \pi + \delta])$ with $0 \leq \delta \leq \pi/2$

- for $\delta = 0$ (totally anisotropic): $c(\widetilde{\mathbf{K}_\varphi}) = 1$, $\forall \varphi$
- for $\delta = \pi/2$ (isotropic): $c(\widetilde{\mathbf{K}_\varphi}) = 0$, $\forall \varphi$
- for $0 < \delta < \pi/2$: $c(\widetilde{\mathbf{K}_\varphi}) = \frac{P_\delta}{Q_\delta}(\cos 2\varphi)$
  with P_δ and Q_δ polynomials of degree 2, only depending on $\operatorname{sinc}(2\delta)$ and $\operatorname{sinc}(4\delta)$
  then $\varphi \mapsto c(\widetilde{\mathbf{K}_\varphi})$ is always critical at $\varphi = \pi/2$

but is $\varphi = \pi/2$ a maximum?

Longuest crest - elementary case (2)

Let $\widetilde{\mathbf{k}} \sim \mathcal{U}([-\delta, \delta] \cup [\pi - \delta, \pi + \delta])$ with $0 \leq \delta \leq \pi/2$



$$\varphi \mapsto c(\widetilde{\mathbf{K}_\varphi}) \text{ for some } \delta \text{ (here } \delta = 0.4\pi)$$

Ccl: longuest crest for $\varphi = \pi/2$, $\perp$ "most probable direction"

Longuest crest - toy model

$\widetilde{\mathbf{k}} = (\cos\Theta, \sin\Theta)$ with $\Theta \sim C_\alpha \,|\cos\theta|^\alpha \, d\theta$

$$\Rightarrow c(\widetilde{\mathbf{K}}_\varphi) = A_\alpha - B_\alpha \,(\varphi - \pi/2) + o(\varphi - \pi/2)$$

with $A_\alpha = c(\widetilde{\mathbf{K}}_{\pi/2})$, $B_\alpha > 0$, for any $\alpha > 0$

Ccl: longuest crest for $\varphi = \pi/2$, $\perp$ most probable direction

Take home message

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Generic procedure:

- X any Gaussian field on $\mathbb{R}^d$, stat. centered, unit variance
- Bochner's thm: $\mathbb{E}(X(0)X(t)) = \int_{\mathbb{R}^d} e^{it\cdot\lambda} d\mu(\lambda)$ with μ probability measure on $\mathbb{R}^d$
- take $\mathbf{k}$ a random vector in $\mathbb{R}^d$ with distribuion μ

X is a Gaussian wave associated with $\mathbf{k}$

Take home work

- study $\varphi \mapsto c(\mathbf{K}_\varphi)$ whatever the distribution of Θ
- compute variance of nodal lines length in Berry's anisotropic planar case
  - Berry's cancellation phenomenon in anisotropic frame?
  - variation of the constant before the leading term
- study second order properties of expected measures of level sets in general anisotropic framework
- visit again arithmetic waves with anisotropic asymptotic spectral measure

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Thank you for your attention