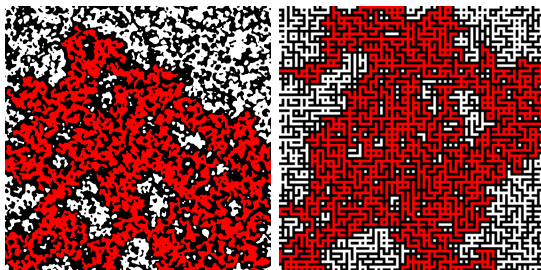
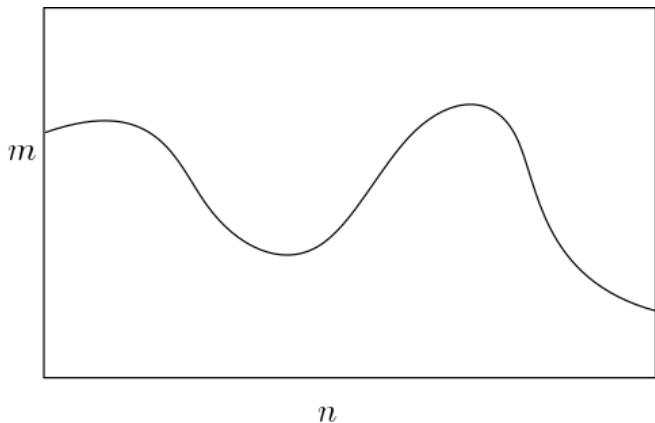


# Percolation and random nodal lines

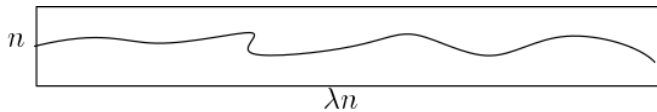
Random waves in Oxford- 18-22 June 2018



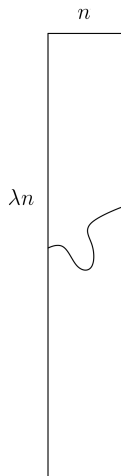
Damien Gayet (Institut Fourier, Grenoble)  
joint work with  
Vincent Beffara (Institut Fourier, Grenoble)



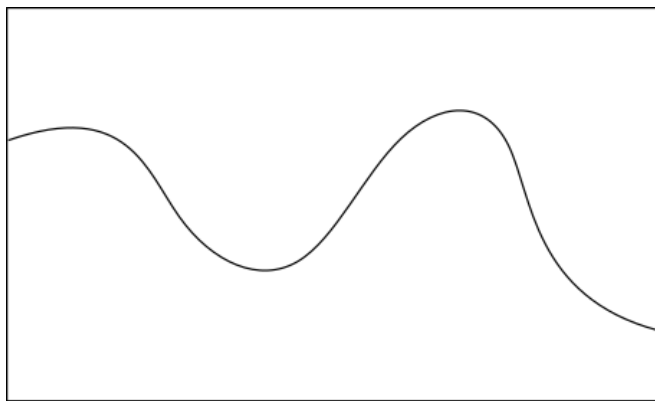
$$\liminf_{n,m \rightarrow \infty} \text{Prob} > c > 0?$$



$$\text{Prob} \xrightarrow{n, \lambda \rightarrow \infty} 0$$



$$\text{Prob} \xrightarrow{n, \lambda \rightarrow \infty} 1$$



$nR$

$$\liminf_{n \rightarrow \infty} \text{Prob} \geq c > 0 ?$$

# Squares

$$\text{Prob} \left[ \begin{array}{|c|} \hline \text{Diagram 1} \\ \hline \end{array} \right] + \text{Prob} \left[ \begin{array}{|c|} \hline \text{Diagram 2} \\ \hline \end{array} \right] = 1$$

The image shows a mathematical equation involving two square diagrams. The first square contains a black curve that starts at the bottom-left corner, rises to a peak, dips to a local minimum, and then rises again to end at the top-right corner. The second square contains a red curve that starts at the top edge, dips to a local minimum, rises to a local maximum, and then dips again to end at the bottom edge. The equation states that the sum of the probabilities of these two events is equal to 1.

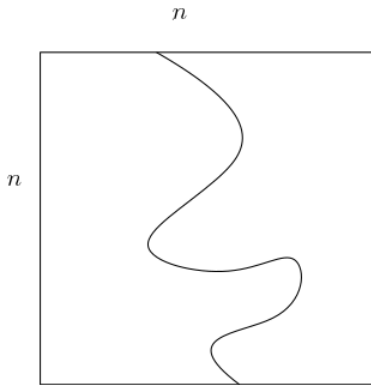
# Squares

$$\text{Prob} \left[ \text{Square with black wavy line} \right] + \text{Prob} \left[ \text{Square with red wavy line} \right] = 1$$

With

- ▶ symmetry between  $+$  and  $-$
- ▶ symmetry between  $x_1$  and  $x_2$

then both probabilities are equal...



Prob =  $1/2$ .

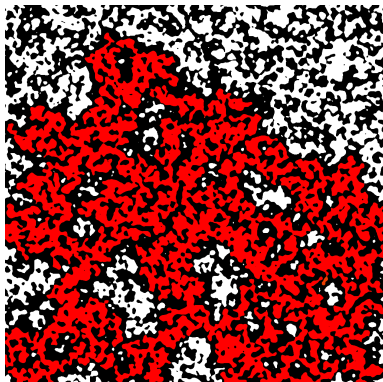




Bond percolation on  $\mathbb{Z}^2$ .

**Theorem (Russo, Seymour-Welsh 1978)** Let  $R \subset \mathbb{R}^2$ .  
Then there exists  $c > 0$ ,

$$\liminf_{n \rightarrow \infty} \text{Prob}(\text{positive crossing of } nR) > c.$$



**Question:** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a random smooth function and fix  $R \subset \mathbb{R}^2$ . Does it exist  $c > 0$ ,

$$\liminf_{n \rightarrow \infty} \text{Prob} \left( \{f > 0\} \text{ crosses } nR \right) > c?$$

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be

- ▶ a centered Gaussian field
- ▶ with symmetric covariant function

$$e(x, y) := E(f(x)f(y)) = k(\|x - y\|).$$

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be

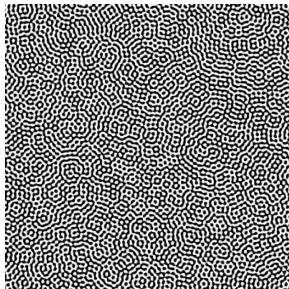
- ▶ a centered Gaussian field
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## Two universal models

- ▶ The random wave model (RW) (Riemannian)
- ▶ The Bargmann-Fock model (algebraic)

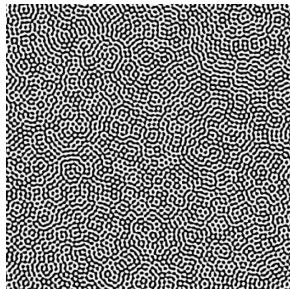
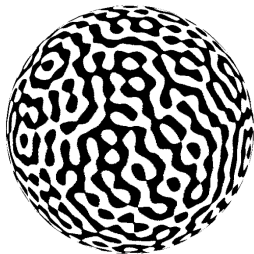
# The random wave model



Barnett, Bogomolny-Schmidt

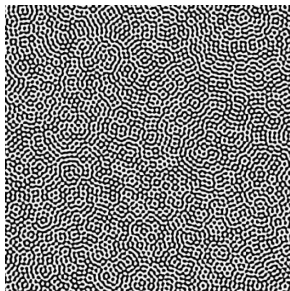
► 
$$g(r, \theta) = \sum_{m=-\infty}^{\infty} a_m J_{|m|}(r) e^{im\theta}$$

# The random wave model



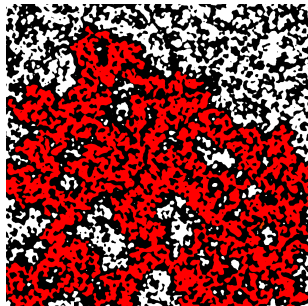
Barnett, Bogomolny-Schmidt

- ▶  $g(r, \theta) = \sum_{m=-\infty}^{\infty} a_m J_{|m|}(r) e^{im\theta}$
- ▶ limit model for the rescaled **spherical harmonics**
- ▶ (and more - universal from Riemannian manifolds).



**Conjecture** (Bogomolny-Schmidt 2007) RSW for this model.

# The Bargmann-Fock model

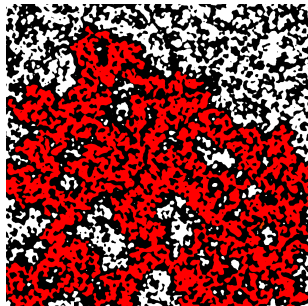


Beffara

► 
$$f(x_1, x_2) = \sum_{i,j=0}^{\infty} a_{ij} \frac{x_1^i x_2^j}{\sqrt{i!j!}}$$

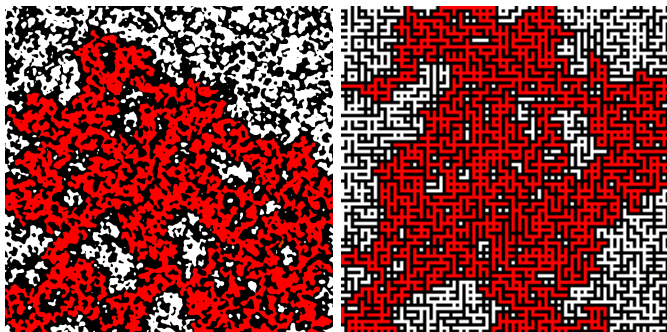


# The Bargmann-Fock model



Nastasescu - Beffara

- ▶  $f(x_1, x_2) = \sum_{i,j=0}^{\infty} a_{ij} \frac{x_1^i x_2^j}{\sqrt{i!j!}}$
- ▶ is the limit for the rescaled **polynomials** for complex Fubini-Study (Kostlan) measure.
- ▶ (and more - universal from algebraic varieties).

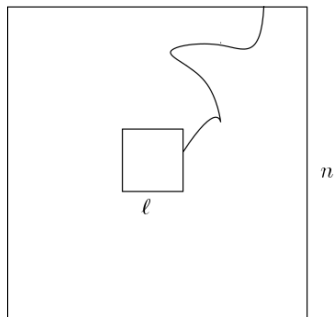


**Theorem** (Beffara-G 2016) RSW holds for Bargmann-Fock:  
for any rectangle  $R$ , there exists  $c > 0$  such that

$$\liminf_{n \rightarrow \infty} \text{Prob} \left( \{f > 0\} \text{ crosses } nR \right) > c.$$

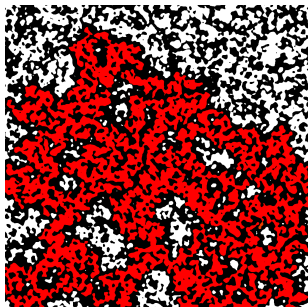
**Remark:** RSW holds for  $0 \leq k(x - y) \leq \|x - y\|^{-325}$

- ▶ Belyaev-Muirhead:  $325 \rightarrow 16$
- ▶ Rivera-Vanneuville:  $325 \rightarrow 4$ .

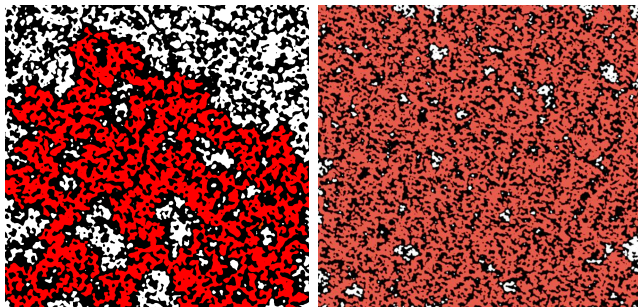


**Corollary (Beffara-G)** For Bargmann-Fock,

$$\text{Prob} < \left(\frac{\ell}{n}\right)^{\alpha > 0}$$



**Corollary (Alexander 1996)** Almost surely there is no infinite component of  $\{f > 0\}$ .



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**Theorem (Rivera-Vanneuille 2017)** For any  $\epsilon > 0$ , almost surely  $\{f > -\epsilon\}$  as an infinite component.



**Theorem** (Belyaev-Muirhead-Wigman 2017) RSW holds for polynomials with the Fubini-Study measure.

# Why Bargmann-Fock and not Random Waves?

- Bargmann-Fock:

$$e(x, y) = \exp(-\|x - y\|^2).$$



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- ▶ Random waves:

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- Bargmann-Fock:

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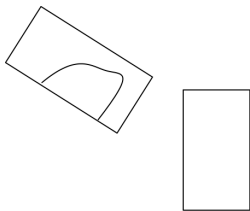
- Random waves:

$$e(x, y) = J_0(\|x - y\|)$$

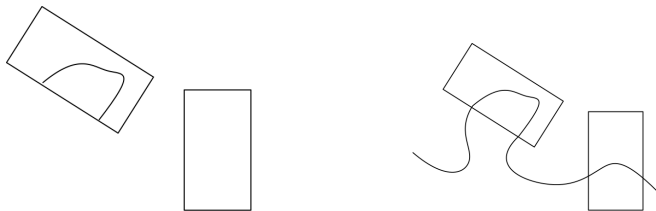
1. oscillating
2. slow decay  $\rightarrow$  strong dependence

Strong decorrelation is not enough...

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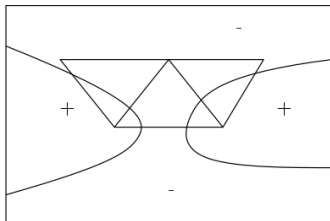


... because of the **Analytic Continuation Phenomenon**.

## Solution : blurring by discretization

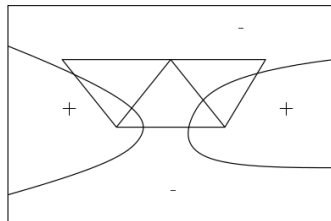
- ▶  $\mathcal{T}$  = triangular lattice,
- ▶  $\mathcal{V}$  = its vertices,
- ▶ sign  $f|_{\mathcal{V}} : \mathcal{V} \rightarrow \{\pm 1\}$ .
- ▶ Site percolation: the edge is positive iff its extremities are.

Is the discretization trustful?



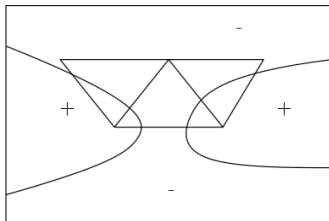


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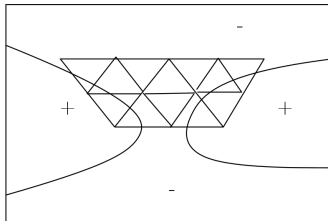


1. If  $\mathcal{T}$  is too coarse, then no.

# Is the discretization trustful?

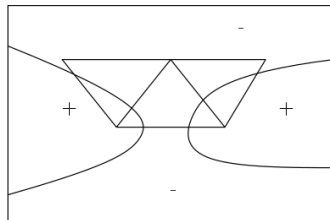


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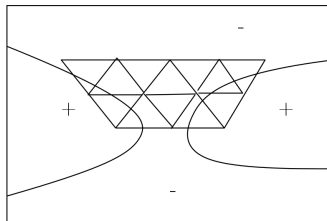


2. If  $\mathcal{T}$  is very thin, then yes, but...

# Is the discretization trustful?



1. If  $\mathcal{T}$  is too coarse, then no.



2. If  $\mathcal{T}$  is very thin, then yes, but... dependence comes back.

## Quantitative good blurring

**Theorem (Beffara-G 2016)** In  $[0, n]^2$ , with high probability,

$$\begin{array}{ccc} \text{continuous} & \text{crossings} & \\ & \Leftrightarrow & \\ \text{discrete} & \text{crossings} & \text{in } \frac{1}{n^9} \mathcal{T}. \end{array}$$

# Quantitative dependence

**Theorem (Beffara-G 2016 - V. Piterbarg 1982)**

$$\max_{\substack{A \text{ crossing in } nR \\ A' \text{ crossing in } nR'}} |\text{Prob}(A \text{ et } A') - \text{Prob } A \text{ Prob } A'|$$

$$\leq$$

$$(\# \text{ vertices in } nR \text{ and } nR')^{8/5} \max_{\substack{x \in nR \\ y \in nR'}} |e(x, y)|^{1/5}.$$

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For our discretization scheme for Bargmann-Fock, on two disjoint  $R$  and  $R'$ , this gives

$$\text{dependence}(nR, nR') \leq n^{50} e^{-n^2/5}$$

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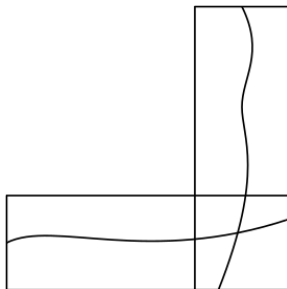
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For our discretization scheme for Bargmann-Fock, on two disjoint  $R$  and  $R'$ , this gives

$$\text{dependence}(nR, nR') \leq n^{50} e^{-n^2/5} \xrightarrow{n \rightarrow \infty} 0.$$

## A crucial tool for RSW

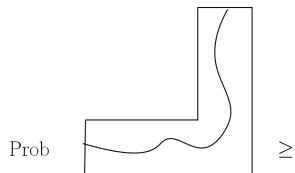


### **FKG (Fortuin-Kasteleyn-Ginibre)**

(crossing = positive crossing). FKG implies

$$\begin{aligned} \text{Prob}(\text{crossing of } R \cap \text{crossing of } R') \\ \geq \\ \text{Prob}(\text{crossing of } R) \cdot \text{Prob}(\text{crossing of } R'). \end{aligned}$$



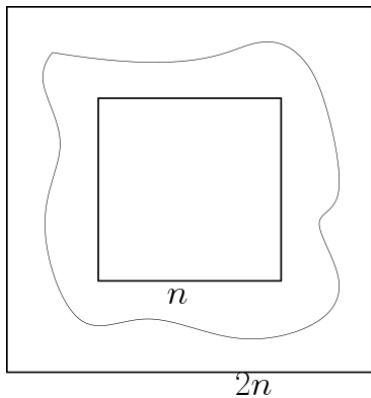


$$\text{Prob} \left[ \text{Diagram 1} \right] \geq \text{Prob} \left[ \text{Diagram 2} \right] \geq$$

The image shows a sequence of two diagrams, each within an L-shaped frame. The first diagram shows a wavy line starting from the left edge, dipping slightly, and then rising to meet the top edge of the vertical part of the L-shape. The second diagram shows a similar wavy line, but it dips more significantly and crosses the bottom edge of the horizontal part of the L-shape before rising to meet the top edge of the vertical part.

$$\begin{aligned}
 & \text{Prob} \left[ \text{Diagram 1} \right] \geq \text{Prob} \left[ \text{Diagram 2} \right] \geq \\
 & \text{Prob} \left[ \text{Diagram 3} \right] \times \text{Prob} \left[ \text{Diagram 4} \right] \\
 & = \text{Prob} (\text{crossing the rectangle})^2
 \end{aligned}$$

The diagrams illustrate a sequence of geometric probability arguments. Diagram 1 is an L-shaped region with a wavy line. Diagram 2 is a similar L-shaped region with a wavy line and a diagonal line. Diagram 3 is a horizontal rectangle with a wavy line. Diagram 4 is a vertical rectangle with a wavy line.



$$\text{Prob} \geq \text{Prob} (\text{crossing the rectangle})^4$$

**Theorem (Loren Pitt 1982)** For Gaussian functions,

$FKG \Leftrightarrow$  positive correlation function.

**Theorem (Tassion 2016)** If we have family of models with

1. FKG
2. uniform crossing of squares
3. uniform asymptotic independence

then we have a uniformly positively bounded RSW.

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- ▶ with high uniform probability the continuous and discrete crossings happen simultaneously
- ▶ the discretization satisfies the three former conditions uniformly in  $n$ .
- ▶ Then Tassion gives a uniform RSW for every scale  $n$ .

## Without positive correlations (without FKG)?

- ▶  $f_B : \mathcal{V} \rightarrow \mathbb{R}$  Gaussian field,

$$\text{sign} f_B = \text{Bernoulli}$$

- ▶  $f : \mathcal{V} \rightarrow \mathbb{R}$  Gaussian field
- ▶ symmetric with strong polynomial decorrelation

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**Theorem (Beffara-G 2017):** For  $\epsilon$  small enough,

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**Theorem (Beffara-G 2017):** For  $\epsilon$  small enough,

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**Remark:** If  $f$  has oscillating correlations, so does  $f_B + \epsilon f$ .

## A smoothed random wave (SRW) model

$$e_{RW}(x, y) = \int_{\mathbb{R}^2} \delta_1(\|\xi\|) e^{i\langle x-y, \xi \rangle} d\xi.$$

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$$e_{RW}(x, y) = \int_{\mathbb{R}^2} \delta_1(\|\xi\|) e^{i\langle x-y, \xi \rangle} d\xi.$$

$$e_{SRW}(x, y) = \int_{\xi \in \mathbb{R}^2} \chi(\|\xi\|) e^{i\langle x-y, \xi \rangle} d\xi.$$

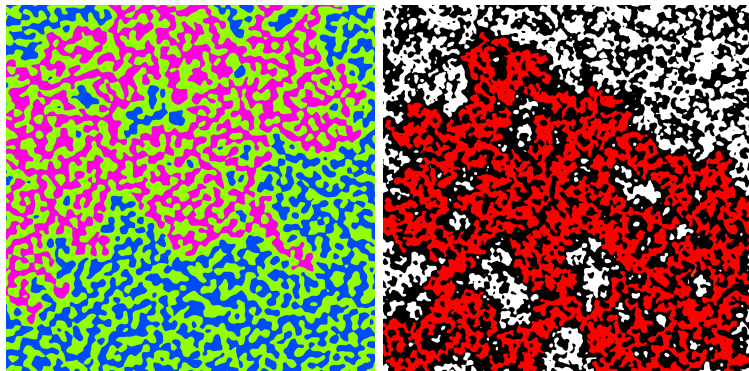
- ▶ If  $\chi$  is smooth with compact support,  $e_{SRW}$  decorrelates strongly.
- ▶ If  $\chi$  is close to  $\delta_1$ , then  $e_{SRW}$  oscillates.

**Corollary:** On a fixed  $\mathcal{V}$ ,

$$f_B + \epsilon f_{SRW}$$

satisfies RSW for  $\epsilon$  small enough.





SRW and BF

## Toy model

**Definition:**  $g : \mathcal{V} \rightarrow \mathbb{R}$  has *finite range*  $\ell$  if

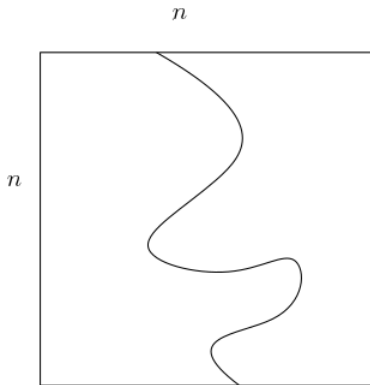
$$\|x - y\| > \ell \Rightarrow e_g(x, y) = 0.$$

## Toy model

**Definition:**  $g : \mathcal{V} \rightarrow \mathbb{R}$  has *finite range*  $\ell$  if

$$\|x - y\| > \ell \Rightarrow e_g(x, y) = 0.$$

With finite range  $\ell$



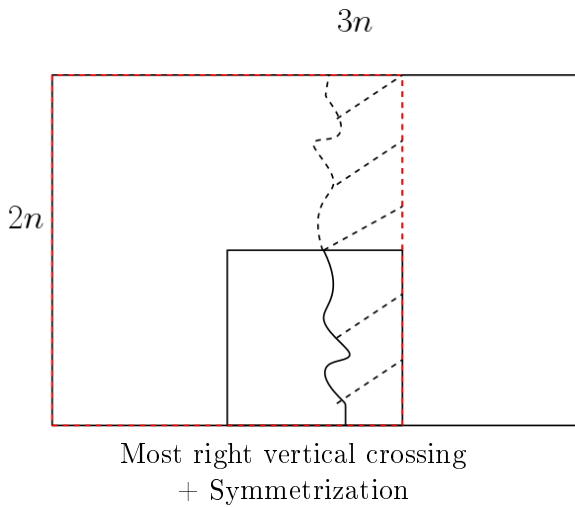
Prob =  $1/2$ .

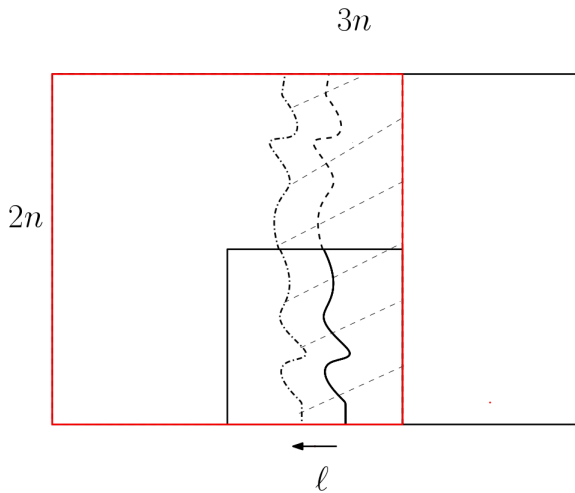
$3n$

$2n$

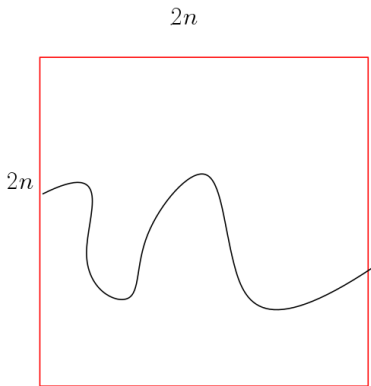


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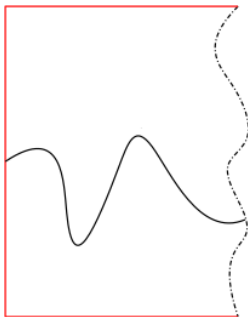


The dependence zone

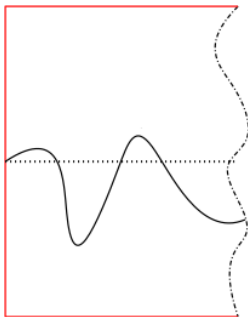


Prob  $\geq 1/2$ .

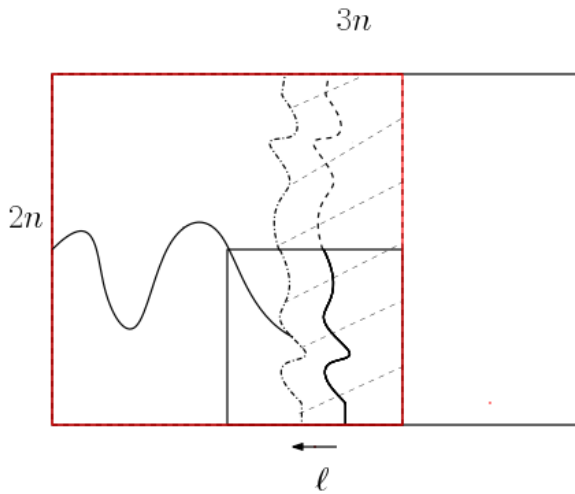




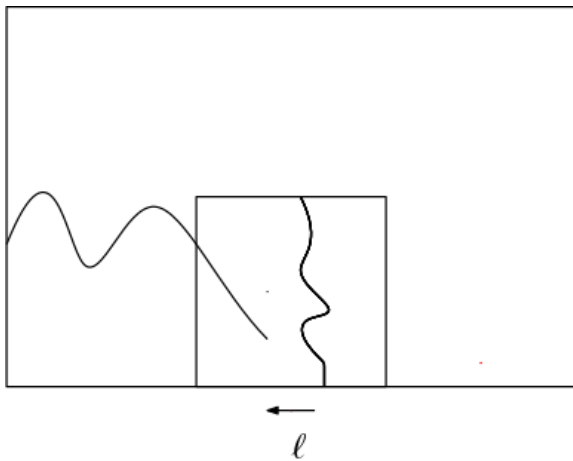
Prob  $\geq 1/2$ .



Prob  $\geq 1/4$ .

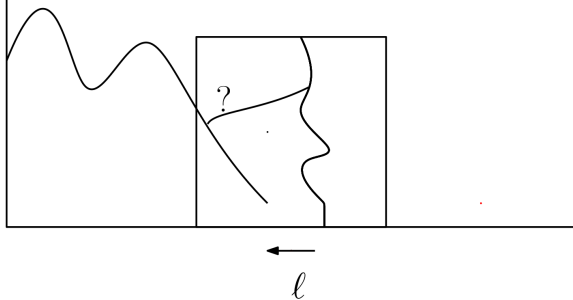


Prob  $\geq 1/8$ .



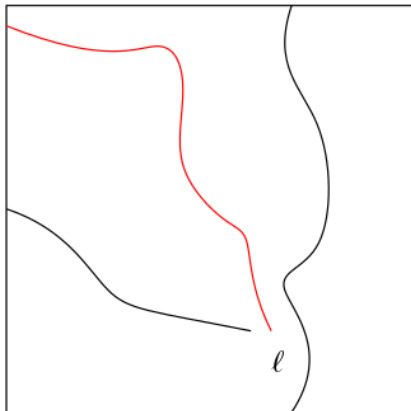
Prob  $\geq 1/8$ .

Is there such a bridge?



If there is no such bridge...

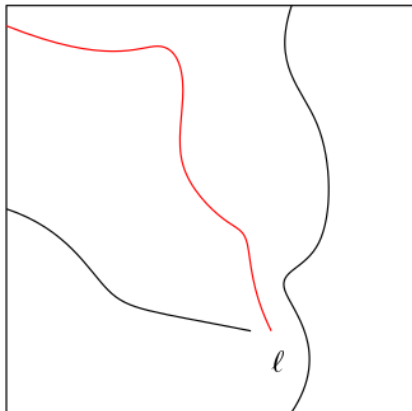
$n$



Negative arm between  $\ell$  and  $n$ .

If there is no such bridge...

$n$

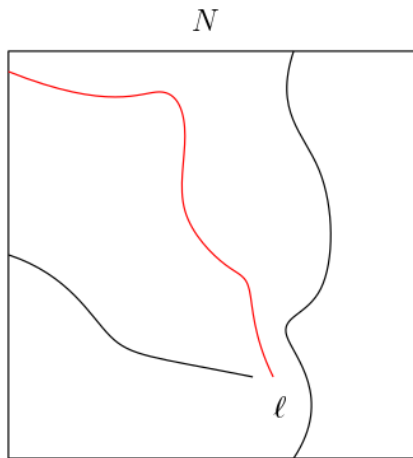


Negative arm between  $\ell$  and  $n$ .

For Bernoulli,

$$\text{Prob} \leq \left(\frac{\ell}{n}\right)^{\alpha>0}$$

Choose  $N$  such that for Bernoulli



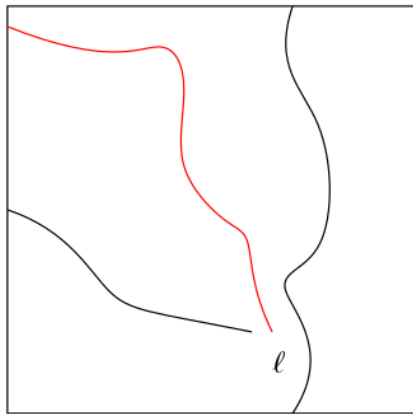
Prob  $\leq 1/32$ .



Then there exists  $\epsilon = \epsilon(N) > 0$  such that for

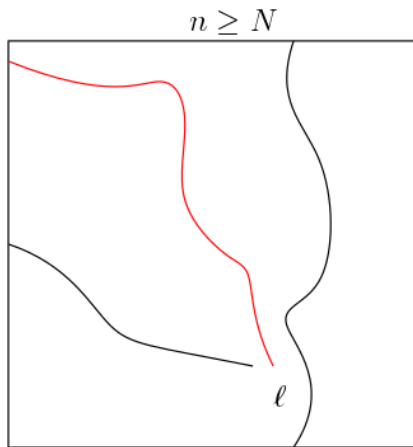
$$f_B + \epsilon f,$$

$N$



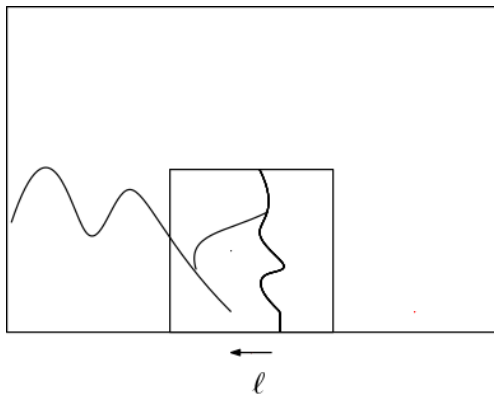
$$\text{Prob} \leq 1/16.$$

For  $f_B + \epsilon f$ ,



Prob  $\leq 1/16$ .

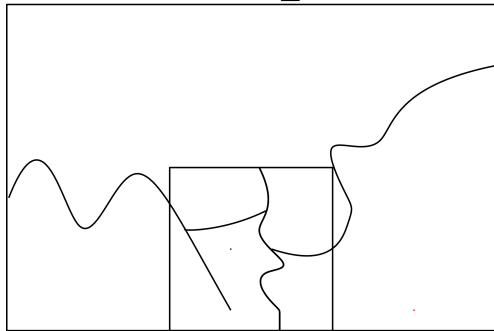
For  $f_B + \epsilon f$  and  $n \geq N$ ,



$$\begin{aligned} \text{Prob} &\geq 1/8 - \text{Prob}(\text{no bridge}) \\ &\geq 1/8 - 1/16 = 1/16. \end{aligned}$$

For  $f_B + \epsilon f$ ,

$$n \geq N$$



$$\text{Prob} \geq 1/256.$$

**Theorem (Beffara-G 2017 V., Piterbarg 1982) :** Let  $f : \mathcal{V} \rightarrow \mathbb{R}$  be a strongly decorrelating Gaussian field. Then

- ▶  $f$  can be coupled with  $g$  with

$$\text{finite range } \sqrt{n} \ll n$$

- ▶ such that with high probability on  $[0, n]^2$ ,

$$\text{sign } f = \text{sign } g.$$