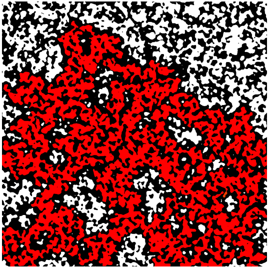
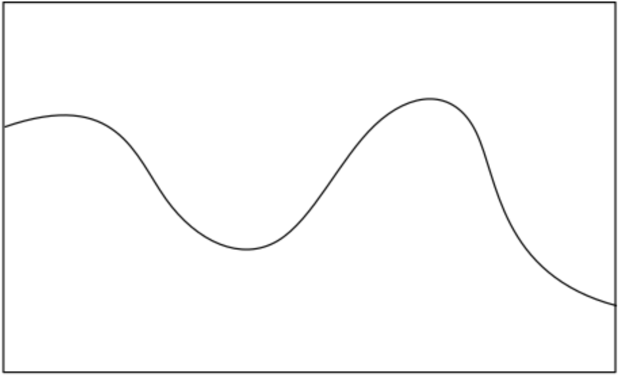
Percolation and random nodal lines

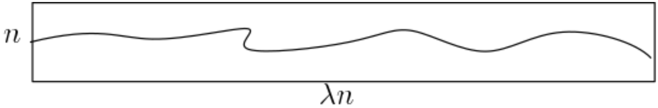
Random waves in Oxford- 18-22 June 2018

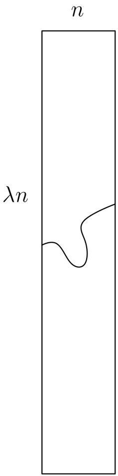
Damien Gayet (Institut Fourier, Grenoble)
joint work with
Vincent Beffara (Institut Fourier, Grenoble)



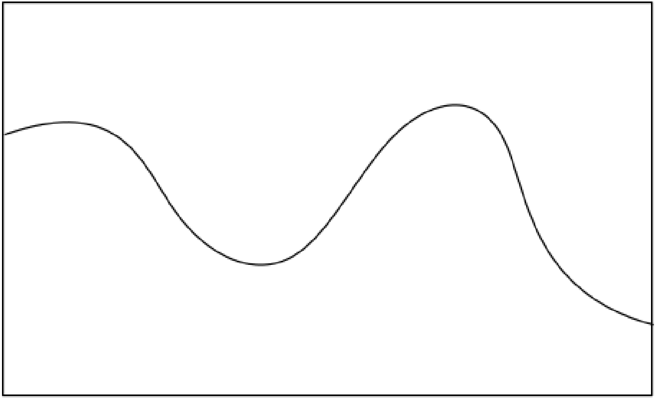
$$\liminf_{n,m\to\infty} \text{Prob} \ > c > 0?$$



$$\text{Prob} \underset{n,\lambda\to\infty}{\longrightarrow} 0$$



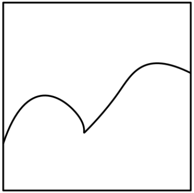
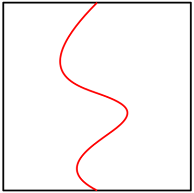
$$\text{Prob} \underset{n,\lambda\to\infty}{\longrightarrow} 1$$



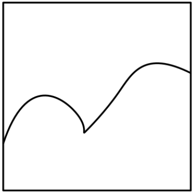
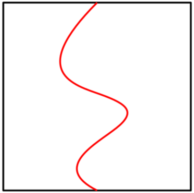
nR

$$\liminf_{n\to\infty} \text{Prob} \;\; \geq c > 0 \text{ ?}$$

Squares

Prob  + Prob  $= 1$

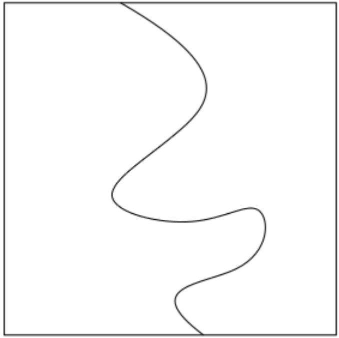
Squares

Prob  + Prob  = 1

With

- symmetry between + and -
- symmetry between x_1 and x_2

then both probabilities are equal...

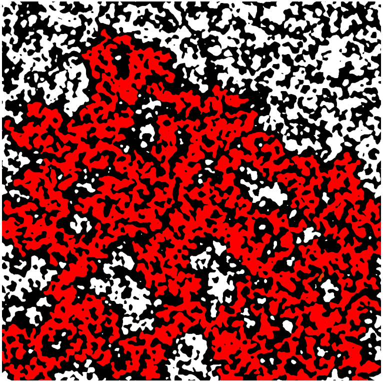


$$\text{Prob } = 1/2.$$

Bond percolation on $\mathbb{Z}^2$.

Theorem (Russo, Seymour-Welsh 1978) Let $R \subset \mathbb{R}^2$.
Then there exists $c > 0$,

$$\liminf_{n\to\infty} \text{Prob (positive crossing of } nR) > c.$$



Question: Let $f : \mathbb{R}^2 \to \mathbb{R}$ a be random smooth function and fix $R \subset \mathbb{R}^2$. Does it exist $c > 0$,

$$\liminf_{n\to\infty} \text{Prob}\ \big(\{f > 0\} \text{ crosses } nR\big) > c?$$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be

- a centered Gaussian field
- with symmetric covariant function

$$e(x, y) := E(f(x)f(y)) = k(\|x - y\|).$$

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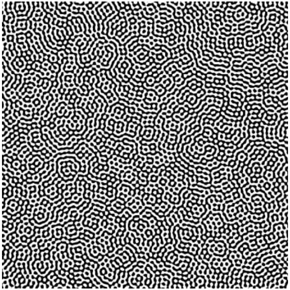
Two universal models

- The random wave model (RW) (Riemannian)
- The Bargmann-Fock model (algebraic)

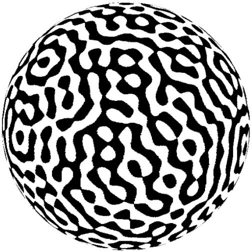
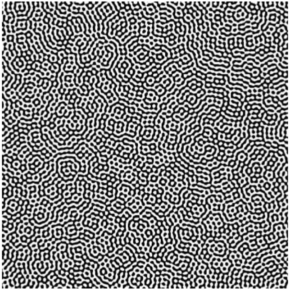
The random wave model



Barnett, Bogomolny-Schmidt

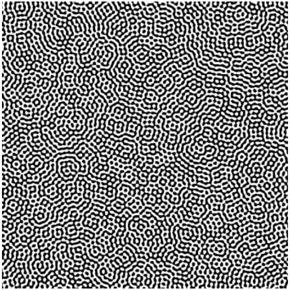
- $g(r, \theta) = \sum_{m=-\infty}^{\infty} a_m J_{|m|}(r) e^{im\theta}$

The random wave model

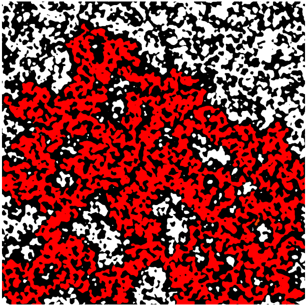
Barnett, Bogomolny-Schmidt

- $g(r,\theta) = \sum_{m=-\infty}^{\infty} a_m J_{|m|}(r) e^{im\theta}$
- limit model for the rescaled **spherical harmonics**
- (and more - universal from Riemannian manifolds).



Conjecture (Bogomolny-Schmidt 2007) RSW for this model.

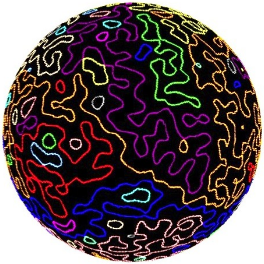
The Bargmann-Fock model

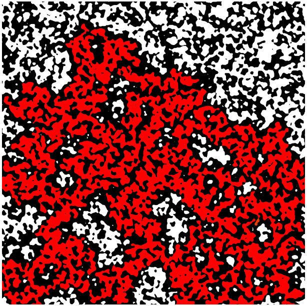


Beffara

- $f(x_1, x_2) = \sum_{i,j=0}^{\infty} a_{ij} \frac{x_1^i x_2^j}{\sqrt{i!j!}}$

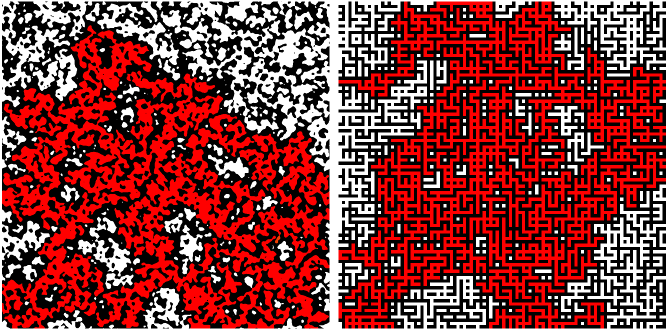
The Bargmann-Fock model





Nastasescu - Beffara

- $f(x_1, x_2) = \sum_{i,j=0}^{\infty} a_{ij} \frac{x_1^i x_2^j}{\sqrt{i!j!}}$
- is the limit for the rescaled **polynomials** for complex Fubini-Study (Kostlan) measure.
- (and more - universal from algebraic varieties).

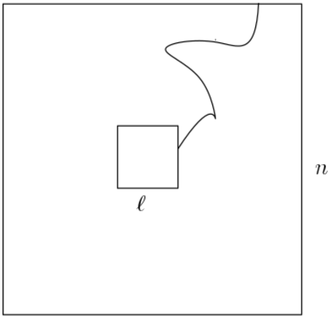


Theorem (Beffara-G 2016) RSW holds for Bargmann-Fock: for any rectangle R, there exists $c > 0$ such that

$$\liminf_{n\to\infty} \text{Prob}\Big(\{f > 0\} \text{ crosses } nR\Big) > c.$$

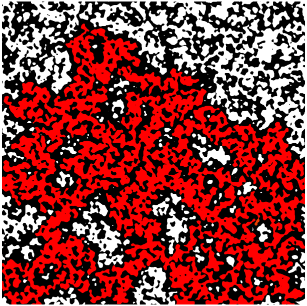
Remark: RSW holds for $0 \leq k(x-y) \leq \|x-y\|^{-325}$

- Belyaev-Muirhead: $325 \to 16$
- Rivera-Vanneuville: $325 \to 4$.

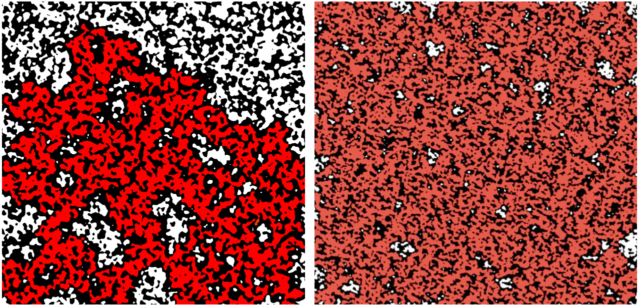


Corollary (Beffara-G) For Bargmann-Fock,

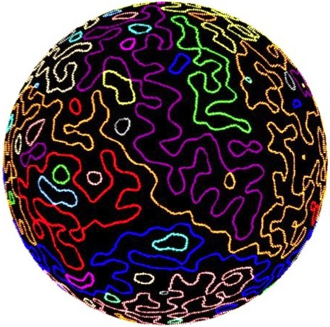
$$\text{Prob } < \left(\frac{\ell}{n}\right)^{\alpha>0}$$



Corollary (Alexander 1996) Almost surely there is no infinite component of $\{f > 0\}$.



Corollary (Alexander 1996) Almost surely there is no infinite component of $\{f > 0\}$.
Theorem (Rivera-Vanneuville 2017) For any $\epsilon > 0$, almost surely $\{f > -\epsilon\}$ as an infinite component.



Theorem (Belyaev-Muirhead-Wigman 2017) RSW holds for polynomials with the Fubini-Studi measure.

Why Bargmann-Fock and not Random Waves?

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- Random waves:

$$e(x,y) = J_0(\|x-y\|)$$

  1. oscillating
  2. slow decay $\rightarrow$ strong dependence

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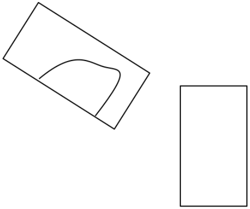
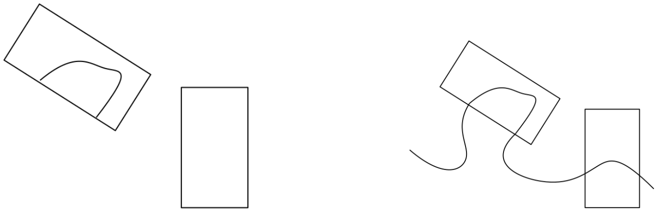


Strong decorrelation is not enough...

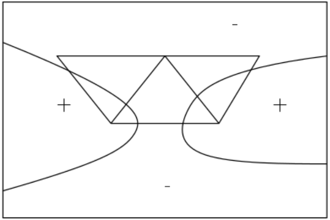


... because of the Analytic Continuation Phenomenon.

Solution : blurring by discretization

- $\mathcal{T}$ = triangular lattice,
- $\mathcal{V}$ = its vertices,
- sign $f_{|\mathcal{V}} : \mathcal{V} \to \{\pm 1\}$.
- Site percolation: the edge is positive iff its extremities are.

Is the discretization trustful?



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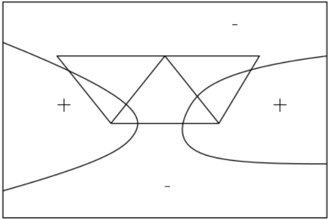

1. If $\mathcal{T}$ is too coarse, then no.

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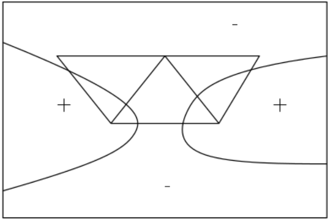

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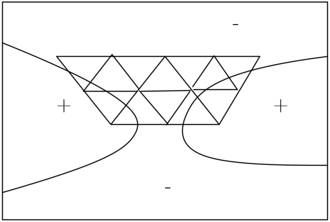

2. If $\mathcal{T}$ is very thin, then yes, but...

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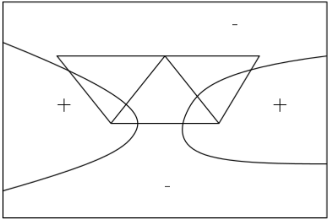

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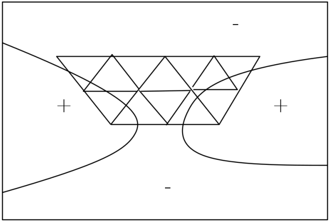

2. If $\mathcal{T}$ is very thin, then yes, but... dependence comes back.

Quantitative good blurring

Theorem (Beffara-G 2016) In $[0,n]^2$, with high probability,

$$\text{continuous} \quad \text{crossings} \iff \text{discrete} \quad \text{crossings} \quad \text{in } \frac{1}{n^9}\mathcal{T}.$$

Quantitative dependence

Theorem (Beffara-G 2016 - V. Piterbarg 1982)

$$\max_{\substack{A \text{ crossing in } nR \\ A' \text{ crossing in } nR'}} |\text{Prob } (A \text{ et } A') - \text{Prob } A \text{ Prob } A'|$$

$$\leq$$

$$(\# \text{ vertices in } nR \text{ and } nR')^{8/5} \max_{\substack{x \in nR \\ y \in nR'}} |e(x,y)|^{1/5}.$$

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For our discretization scheme for Bargmann-Fock, on two disjoint R and R', this gives

$$\text{dependence}(nR, nR') \leq n^{50} e^{-n^2/5}$$

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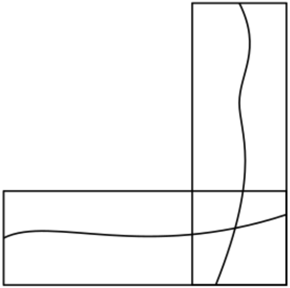
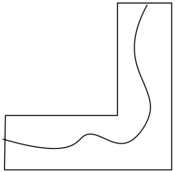
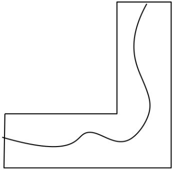
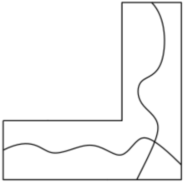
A crucial tool for RSW

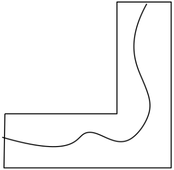
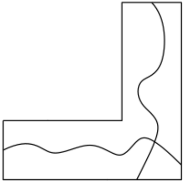


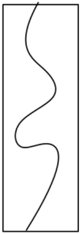
FKG (Fortuin-Kasteleyn-Ginibre)
(crossing = positive crossing). FKG implies

$$\begin{array}{ccc} \text{Prob}\big(\text{crossing of } R & \cap & \text{crossing of } R'\big) \\ & \geq & \\ \text{Prob}(\text{crossing of } R) & . & \text{Prob}(\text{crossing of } R'). \end{array}$$

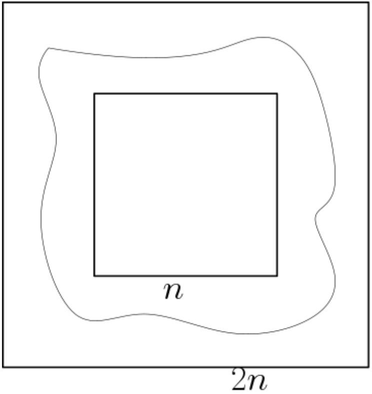
Prob  $\geq$

Prob  $\geq$ Prob  $\geq$

Prob  $\geq$ Prob  $\geq$

Prob  $\times$ Prob 

$= \text{Prob (crossing the rectangle)}^2$



Prob $\geq$ Prob (crossing the rectangle)^4

Theorem (Loren Pitt 1982) For Gaussian functions,

$$FKG \Leftrightarrow \text{positive correlation function.}$$

Theorem (Tassion 2016) If we have family of models with

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2. uniform crossing of squares
3. uniform asymptotic independence

then we have a uniformly positively bounded RSW.

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In conclusion:

- for every n we discretize on $[0,n]^2$
- with high uniform probability the continuous and discrete crossings happen simultaneously
- the discretization satisfies the three former conditions uniformly in n.
- Then Tassion gives a uniform RSW for every scale n.

Without positive correlations (without FKG)?

- $f_B : \mathcal{V} \to \mathbb{R}$ Gaussian field,

$$\operatorname{sign} f_B = \text{Bernoulli}$$

- $f : \mathcal{V} \to \mathbb{R}$ Gaussian field
- symmetric with strong polynomial decorrelation

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Remark: If f has oscillating correlations, so does $f_B + \epsilon f$.

A smoothed random wave (SRW) model

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$$e_{RW}(x,y) = \int_{\mathbb{R}^2} \delta_1(\|\xi\|) e^{i\langle x-y,\xi\rangle} d\xi.$$

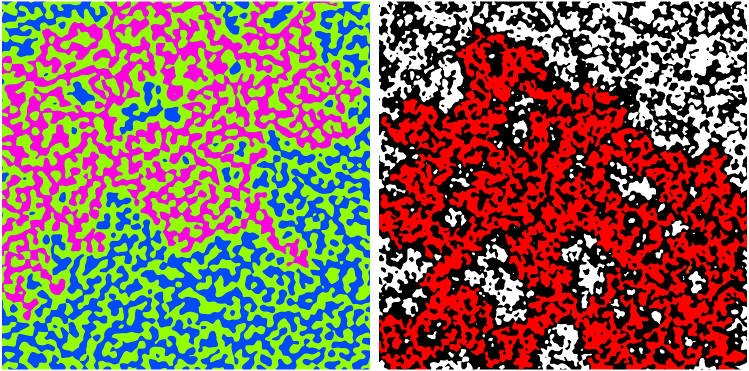
$$e_{SRW}(x,y) = \int_{\xi\in\mathbb{R}^2} \chi(\|\xi\|) e^{i\langle x-y,\xi\rangle} d\xi.$$

- If χ is smooth with compact support, e_{SRW} decorrelates strongly.
- If χ is close to δ_1, then e_{SRW} oscillates.

Corollary: On a fixed $\mathcal{V}$,

$$f_B + \epsilon f_{SRW}$$

satisfies RSW for ϵ small enough.



SRW and BF

Toy model

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$$\|x - y\| > \ell \Rightarrow e_g(x, y) = 0.$$

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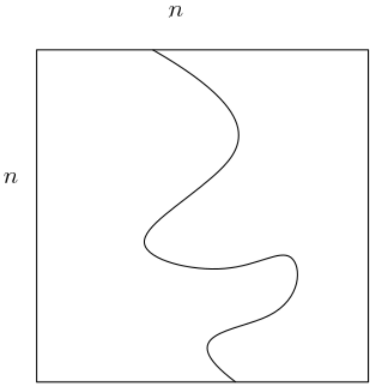
With finite range ℓ



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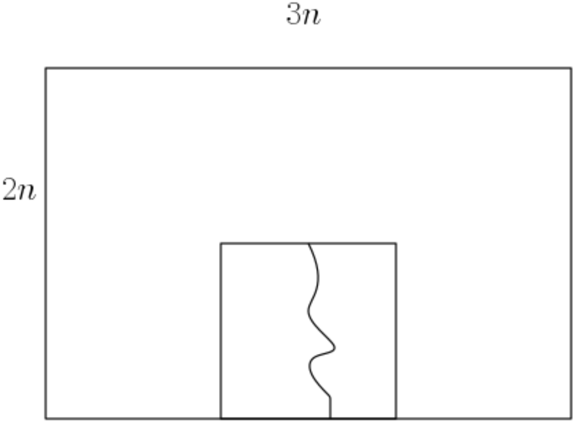

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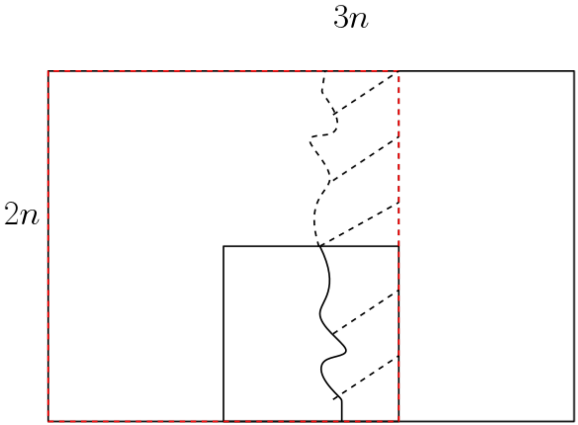
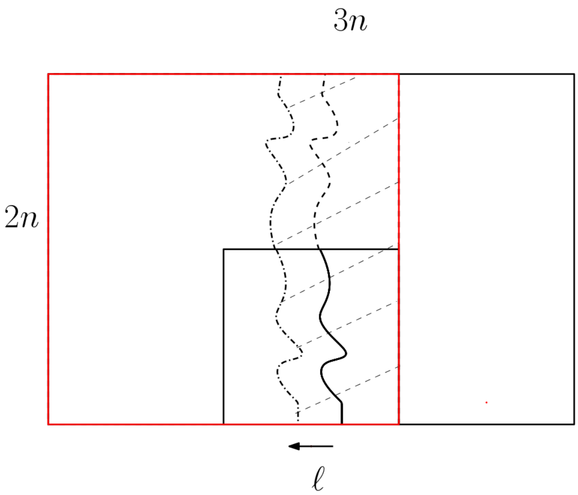
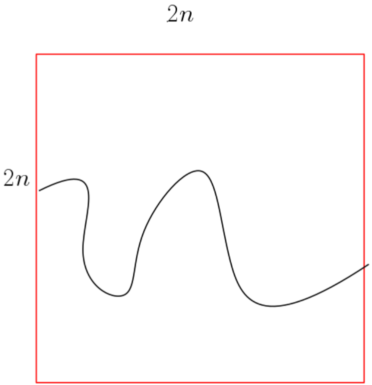


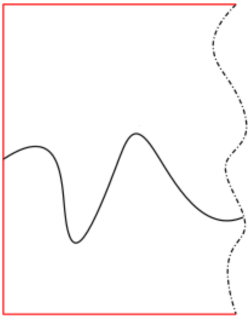
Most right vertical crossing
+ Symmetrization



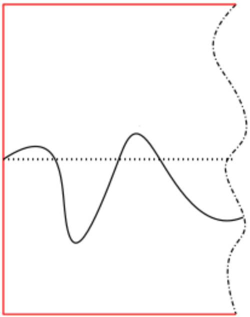
The dependence zone



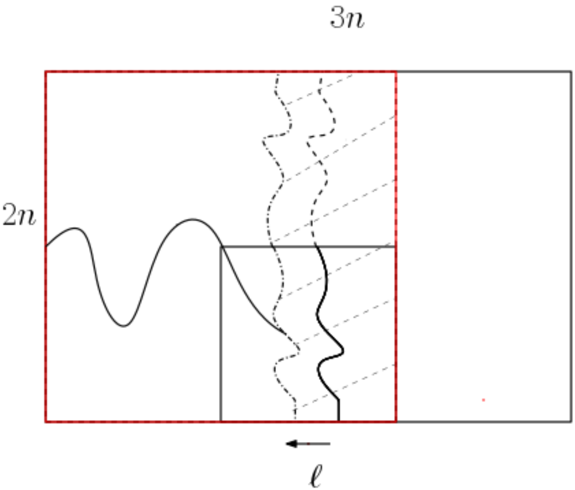
Prob $\geq 1/2$.



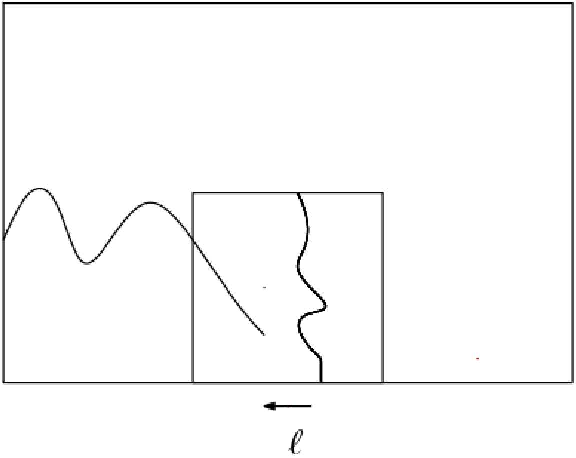
Prob $\geq 1/2$.



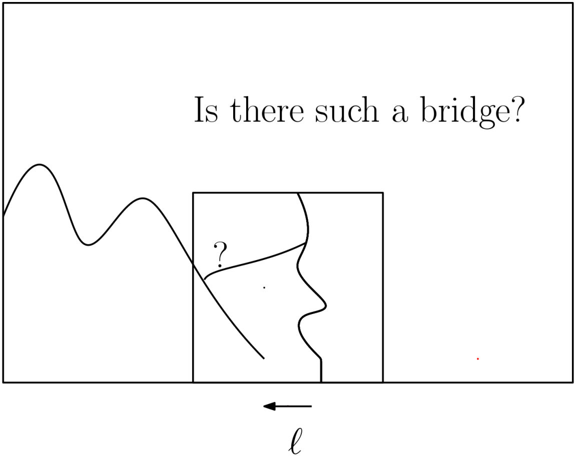
Prob $\geq 1/4$.



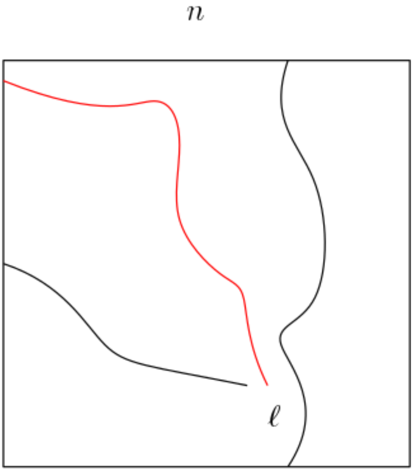
Prob $\geq 1/8$.



Prob $\geq 1/8$.


Is there such a bridge?
?
ℓ

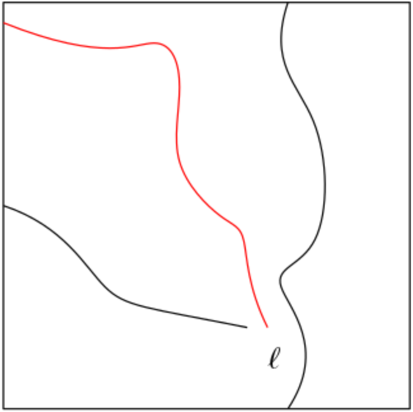
If there is no such bridge...



Negative arm between ℓ and n.

If there is no such bridge...

n

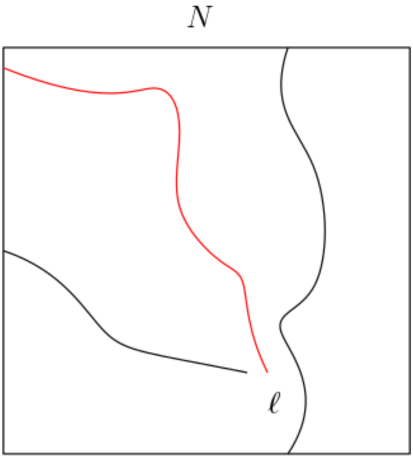


Negative arm between ℓ and n.

For Bernoulli,

$$\text{Prob} \leq \left(\frac{\ell}{n}\right)^{\alpha>0}$$

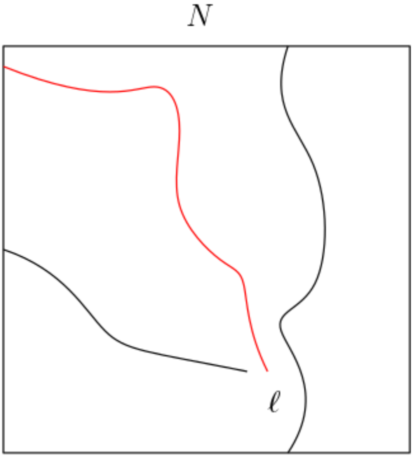
Choose N such that for Bernoulli



Prob $\leq 1/32$.

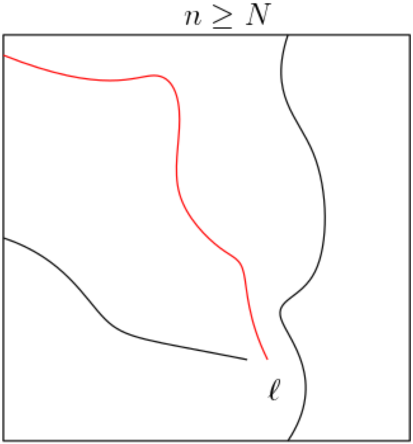
Then there exists $\epsilon = \epsilon(N) > 0$ such that for

$$f_B + \epsilon f,$$



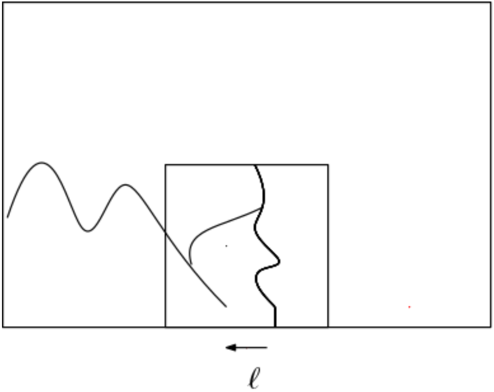
$$\text{Prob} \leq 1/16.$$

For $f_B + \epsilon f$,



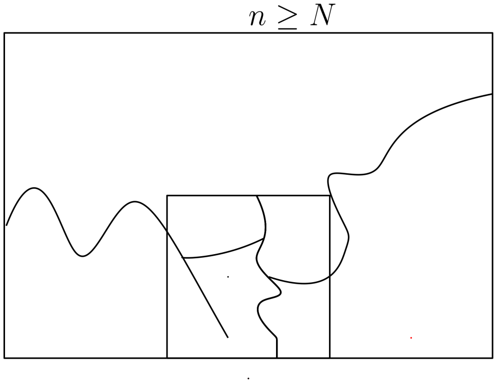
Prob $\leq 1/16$.

For $f_B + \epsilon f$ and $n \geq N$,



$$\begin{aligned} \text{Prob} &\geq 1/8 - \text{Prob(no bridge)} \\ &\geq 1/8 - 1/16 = 1/16. \end{aligned}$$

For $f_B + \epsilon f$,



Prob $\geq 1/256$.

Theorem (Beffara-G 2017 V., Piterbarg 1982) : Let $f : \mathcal{V} \to \mathbb{R}$ be a strongly decorrelating Gaussian field. Then

- f can be coupled with g with

$$\text{finite range } \sqrt{n} \ll n$$

- such that with high probability on $[0, n]^2$,

$$\text{sign } f = \text{sign } g.$$