Variation of the Nazarov-Sodin constant for random plane waves and arithmetic random waves

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Random Waves in Oxford

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1. Motivation & Background

General Setup

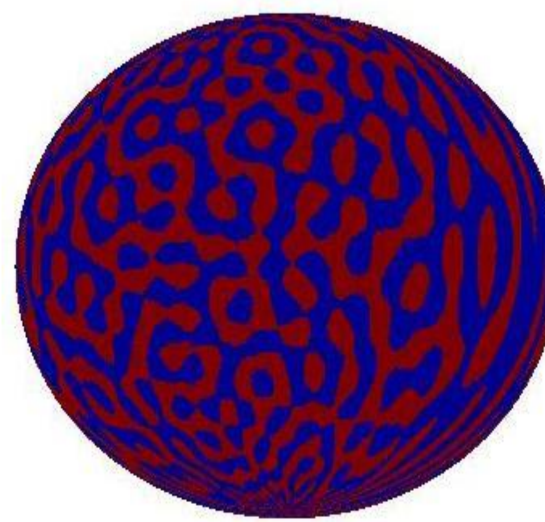
- (M, g) – Compact smooth surface (can generalize higher dimensions)
- Δ Laplace-Beltrami on M
- Eigenfunctions: (boundary condition)

$$\lambda_j \geq 0 \qquad \Delta\varphi_j + \lambda_j\varphi_j = 0$$

- Orthonormal basis of L^2(M,dVol), $\lambda_j \to \infty$

Nodal components & domains

- Nodal set: $Z(\varphi_j) = \varphi_j^{-1}(0)$
- Nodal components: Connected components of $\varphi_j^{-1}(0)$.
- Nodal domains: Connected components of $M \setminus \varphi_j^{-1}(0)$ smooth
- Nodal count: How many components (domains)?



Interesting Questions – non-local

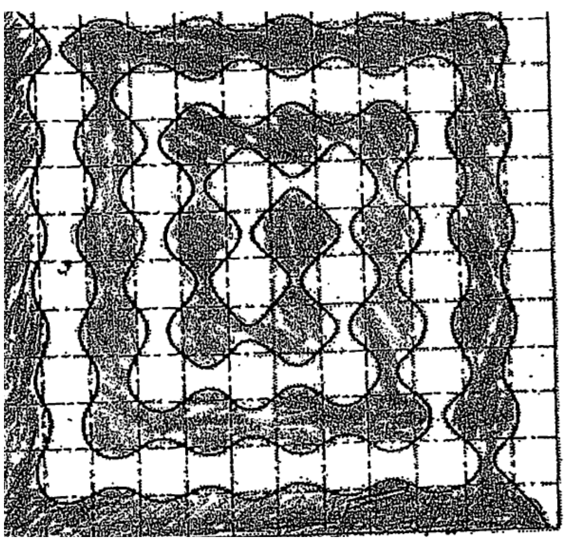
- Nodal count (Nazarov-Sodin `09,`12, `15, Kurlberg-W `14, `17)
- Topology, nesting,geometry(Sarnak-W, Beliaev-W)
- Local: $|Len_{A\cup B}(f)| = |Len_A(f)| + |Len_B(f)|$

$$A \cap B = \emptyset$$

- Semilocality: "Most" of the nodal domains of diameter $\frac{R}{\sqrt{\lambda}}$, R>>0.
- Approximate **locally**

Nodal count (deterministic)

- **Nodal Count.** Courant: $N(\varphi_j) \leq j$
- Pleijel: $\limsup_{j\to\infty} \frac{N(\varphi_j)}{j} \leq 0.691\ldots$
- Constant improved by $3 \cdot 10^{-9}$ (Bourgain)
- No lower bound $N(\varphi_j) \geq 2$



Nodal picture for the square, arbitrarily high energy. A. Stern's thesis, Gottingen, 1925.
Courtesy of P. Sarnak.

Berry's Random Wave Model

- M chaotic. As $\lambda \to \infty$, φ_j "behave randomly" wavenumber $\sqrt{\lambda}$ monochromatic wave $\mathbb{R}^2$

$$u_{\sqrt{\lambda}}(x) = \frac{1}{\sqrt{J}} \Re e \left(\sum_{j=1}^{J} e^{i(\sqrt{\lambda}\langle x, \vartheta_j \rangle + \psi_j)} \right)$$

- Scale invariant, assume $u = u_1$
- Centered Gaussian, covariance

$$\mathrm{E}[u(x) \cdot u(y)] = J_0(|x - y|)$$

- Spectral measure – arc length on unit circle

2. Random Band Limited Functions

Random Band-Limited Functions

- Fix M – smooth n-manifold, $0 \leq \alpha \leq 1$

  $f_\lambda(\mathrm{x}) = \sum_{\alpha\lambda \leq \lambda_j \leq \lambda} a_j \varphi_j(x)$, a_j - N(0,1) i.i.d.

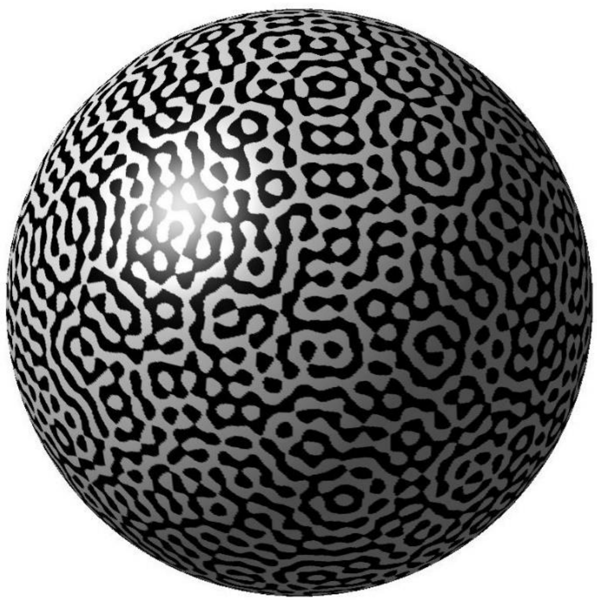
  ($\alpha = 1$ summation over $\lambda - o(\lambda) \leq \lambda_j \leq \lambda$)
- Covariance function

$$\mathbb{E}\left[f_\lambda(x) f_\lambda(y)\right] = \sum \varphi_j(x)\varphi_j(y)$$

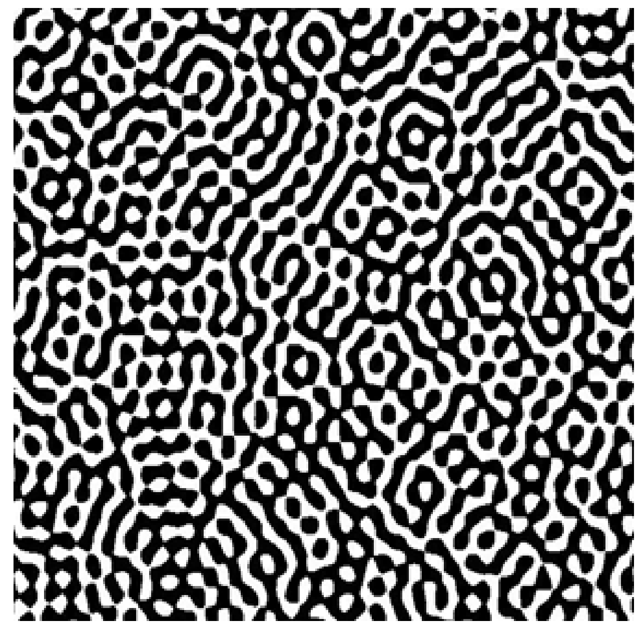
i.e. the spectral projector.

Example 1. Random Spherical harmonics

- $\alpha = 1, M = \mathcal{S}^2$, 2d sphere.
- $\mathbb{E}[T_l(\mathrm{x}) \cdot T_l(\mathrm{y})] = P_l(\cos(d(x,y)))$.
- $P_l(\cos(d)) \approx J_0(ld)$ Legendre fast uniform
- Scales Berry's RWM

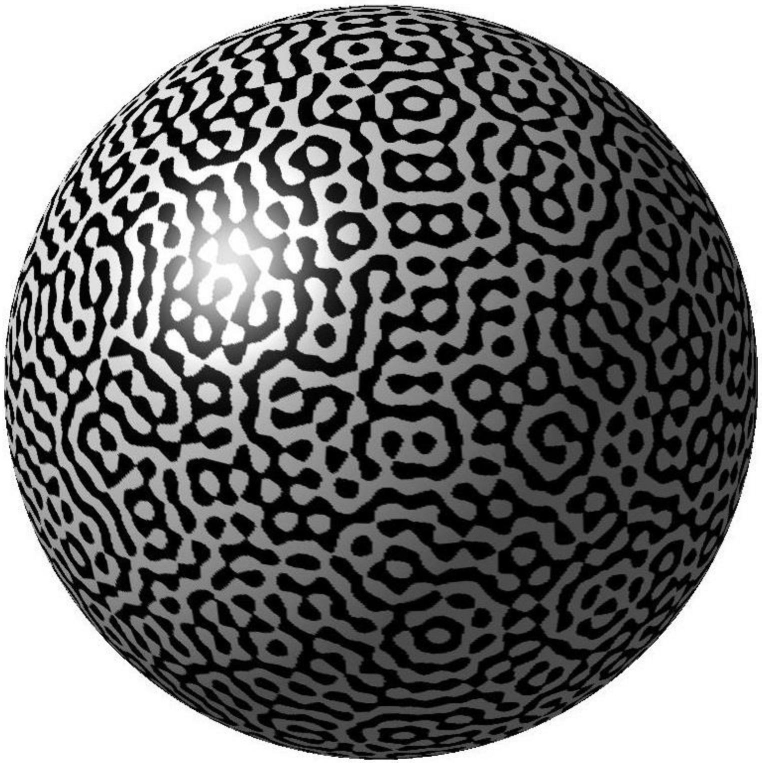


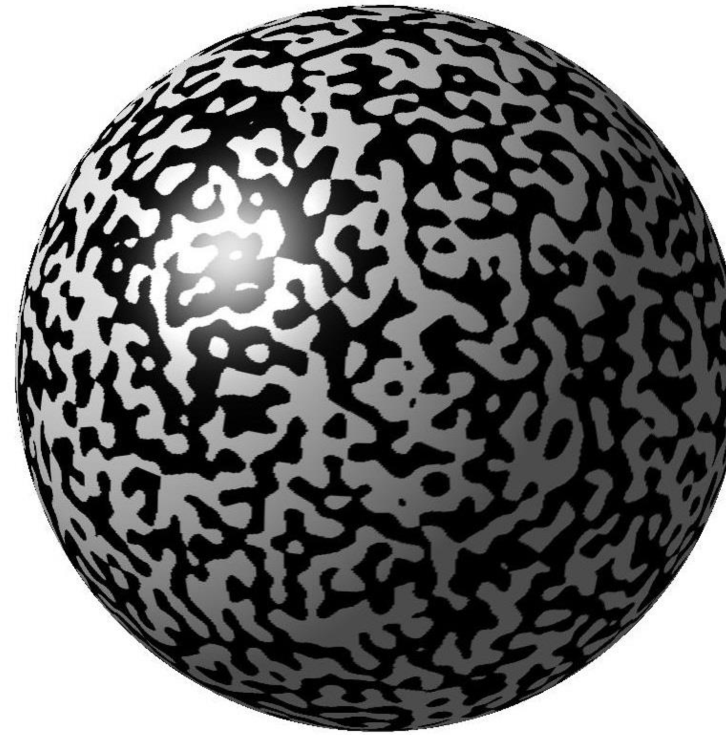
Random spherical harmonics
A. Barnett



RWM

α=1 vs α=0 (Alex Barnett)



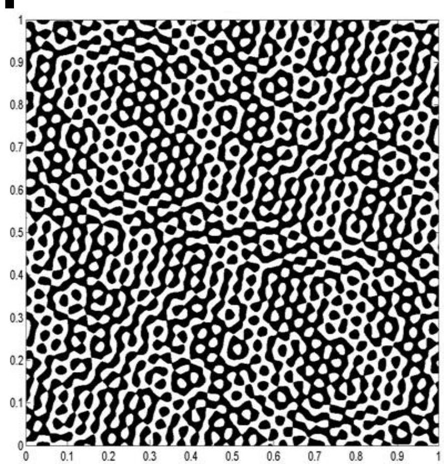


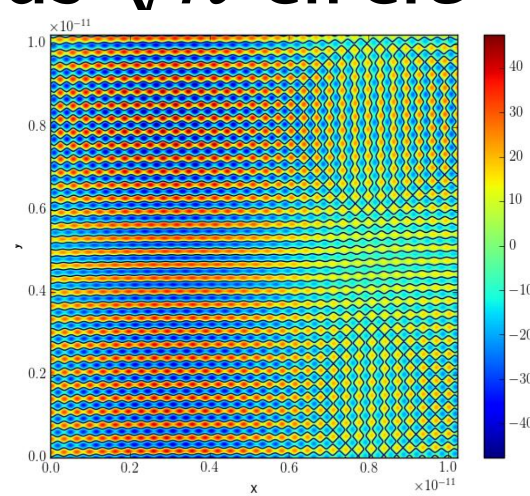
α=1 Random spherical harmonics	α=0 "Real Fubini-Study"

Example 2. Toral eigenfunctions.

- $\mathbb{T} = M = \mathbb{R}^2/_{\mathbb{Z}^2}$
- $f_n(x) = \sum_{\|\mu\|^2=n} a_\mu \cdot e(\langle x, \mu \rangle)$

  a_μ standard Gaussian i.i.d. (save to $a_{-\mu} = \overline{a_\mu}$) "**arithmetic random waves**"
- Summation over $\{\mu \in \mathbb{Z}^2 \colon \|\mu\|^2 = n\}$ lattice points on radius $\sqrt{n}$ circle





More general: limiting ensembles

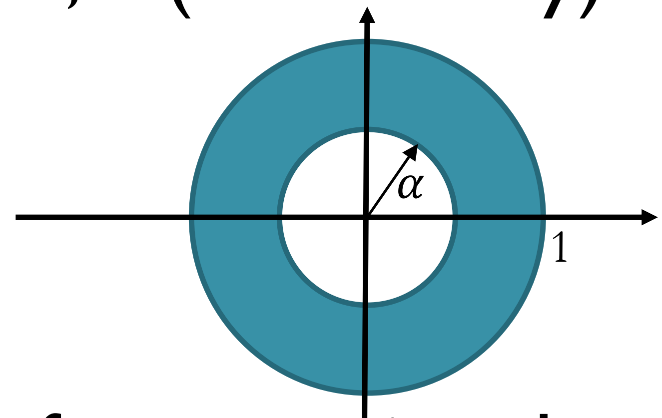
- Natural scaling around any point of M.
- Scaling for covariance (values & derivatives) (classical Hormander, Lax)

$$\mathbb{E}[f_\lambda(\mathrm{x}) \cdot f_\lambda(\mathrm{y})] \approx K_\alpha\left(\sqrt{\lambda} \cdot d(x,y)\right)$$

- $K_\alpha(w) = K_\alpha(\|w\|) = \int_{\alpha \le \|w\| \le 1} e(\langle w, \xi \rangle) d\xi$
- Canzani-Hanin `16 $\alpha = 1$ thin window.
- Define g_∞ on $\mathbb{R}^2$, "clean" covariance $\mathbb{E}[g_\infty(z) \cdot g_\infty(z')] = K_\alpha(\|z - z'\|)$

Limiting ensembles (cont.)

- $\mathbb{E}[g_\infty(z) \cdot g_\infty(z')] = K_\alpha(\|z - z'\|)$
- g_∞ scaling limit f_λ (everywhere)
- g_∞ depends on α, <u>not</u> on M, x (universality)
- Spectral measure



- Relevant: nodal structures of g_∞ restricted on ball $B(R), R \to \infty$.
- E.g. nodal count of domains lying in $B(R)$.

3. Nazarov-Sodin Constant

Scale invariant (Euclidean) case

- $F: \mathbb{R}^2 \to \mathbb{R}$ stationary Gaussian field
- ρ spectral measure of F
- $N(F; R)$ is the number of connected components (domains) of F inside $B(R)$
- Assuming: 1. F ergodic (ρ has no atoms)
  2. F smooth. 3. Non-degeneracy
- Nazarov-Sodin (`12,`15): $c = c_{NS}(\rho) \geq 0$
  $$E[N(F; R)] = c \cdot R^2 + o_{R\to\infty}(R^2)$$
- "Usually" $c > 0$ (support of ρ)

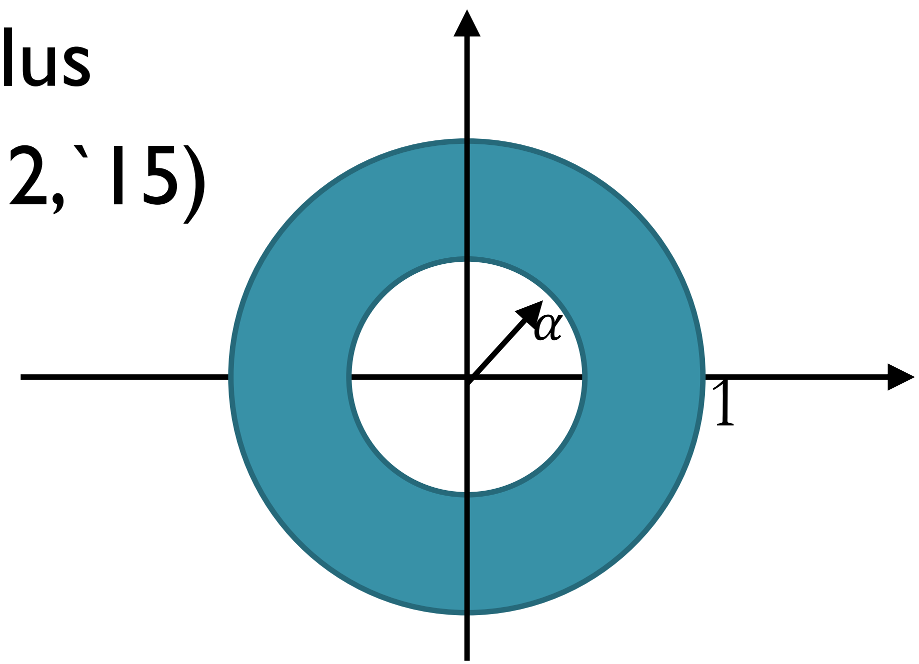
Nodal count band-limited functions

- $c = c_{NS}(\rho),\ \mathrm{E}[N(F;R)] = c \cdot R^2 + o_{R\to\infty}(R^2)$
- Stronger convergence in <u>mean</u> (ergodicity)

$$\mathrm{E}\left[\left|\frac{N(F;R) - c\cdot R^2}{R^2}\right|\right] \to 0$$

- Band-limited functions $c = c_{NS}(\rho_\alpha) > 0$

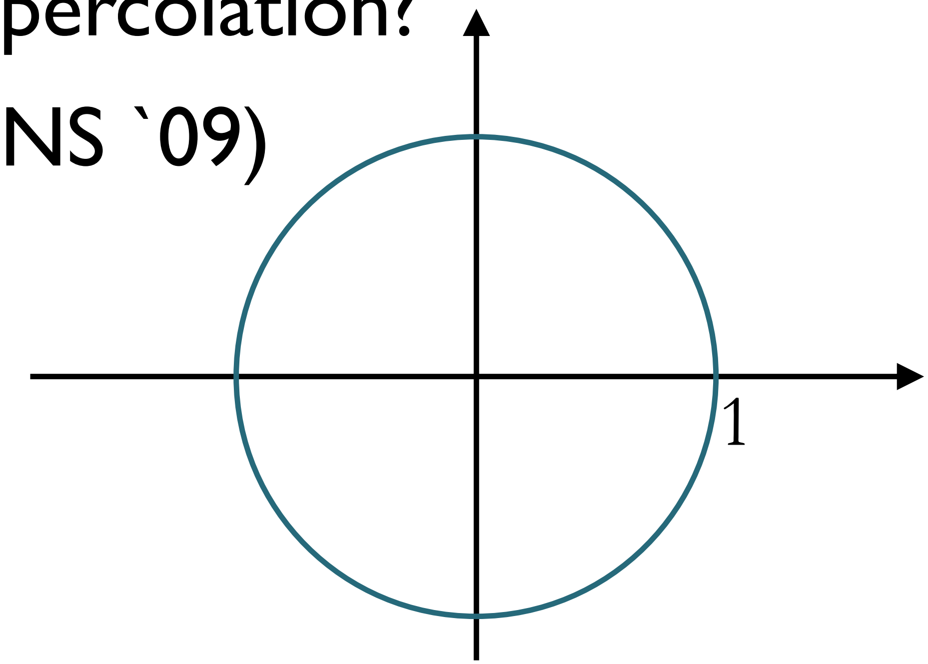
  ρ_α area measure annulus
- $\mathrm{E}[N(f_\lambda)] \sim c \cdot \lambda$ (NS `12,`15)



- Convergence in mean

Random spherical harmonics

- u =Plane monochromatic waves (RWM)
- $E[N(u;R)] \sim c_{RWM} \cdot R^2$, universal NS constant
- $c_{RWM} = c_{NS}\left(\frac{d\theta}{2\pi}\right) > 0$ percolation?
- $E[N(T_l)] \sim c_{RWM} \cdot l^2$ (NS `09)
- Convergence in mean



- Exponential probability concentration

4. Variation on Nazarov-Sodin consant

Generalise NS constant

- Restrict to ρ supported on the unit ball $\mathcal{P}$ (spectral moments), includes band limited case
- Proposition 1(Kurlberg-W): $c = c_{NS}(\rho)$, $\mathrm{E}[N(F;R)] = c \cdot R^2 + O(R), \rho \in \mathcal{P}$ arbitrary, absolute constant (uniform)
- $c_{NS}(\rho)$ bounded (e.g. critical points Kac-Rice)
- No convergence in mean. Can construct examples $\mathrm{E}\left[\frac{|N(F;R) - c \cdot R^2|}{R^2}\right]$ doesn't vanish
- Example: ρ atomic supported at 0. $F \equiv const$, Gaussian $N(F;R) \equiv 0, \quad \Rightarrow c_{NS}(\rho) = 0.$

Variation of NS constant

- Proposition 2 (Kurlberg-W):

  $$d_{NS}(\rho) := \lim_{R\to\infty} \mathrm{E}\left[\frac{|N(F;R)-c\cdot R^2|}{R^2}\right]$$ exists

  (“NS discrepancy functional”) non-uniform, discontinous
- Theorem 1 (Kurlberg-W): $c_{NS}(\rho): \mathcal{P} \to \mathbb{R}_{\geq 0}$ is continuous (weak* topology on $\mathcal{P}$).
- Corollary: $c_{NS}(\rho)$ attains an *interval* $[0, c_0]$ ($\mathcal{P}$ is essential)
- Q: Is it true that $c = c_{RWM}$, *uniquely*?

5. Toral Eigenfunctions

Example 2. Toral eigenfunctions.

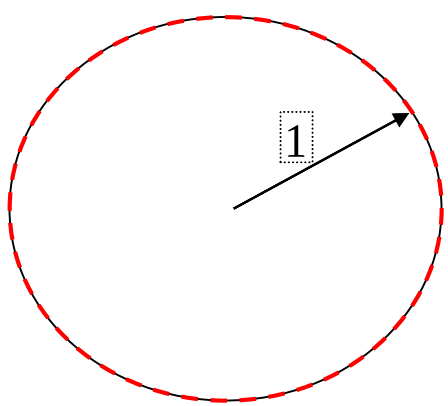
- $\mathbb{T} = M = {}^{\mathbb{R}^2}/_{\mathbb{Z}^2}$
- $f_n(x) = \sum_{\|\mu\|^2=n} a_\mu \cdot e(\langle x, \mu \rangle)$
  a_μ standard Gaussian i.i.d. (save to $a_{-\mu} = \overline{a_\mu}$) "**arithmetic random waves**"
- Summation over $\{\mu \in \mathbb{Z}^2 \colon \|\mu\|^2 = n\}$ lattice points on radius $\sqrt{n}$ circle
- $N(f_n(x))$ nodal count

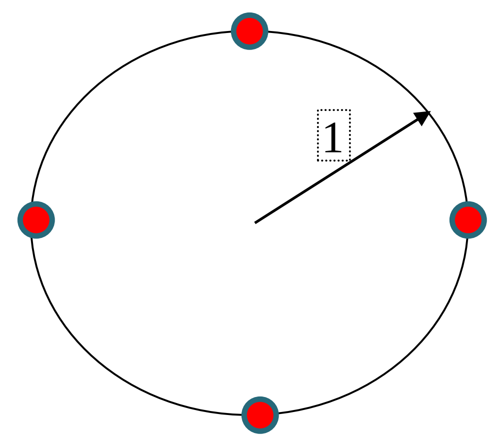
On the 2 squares problem

- $r_2(n) = \#\{(a,b)\epsilon\mathbb{Z}^2 : a^2 + b^2 = n\}$
- On average $r_2(n) \sim c \cdot \sqrt{\log(n)}$ (E. Landau)

Equidistributed generic

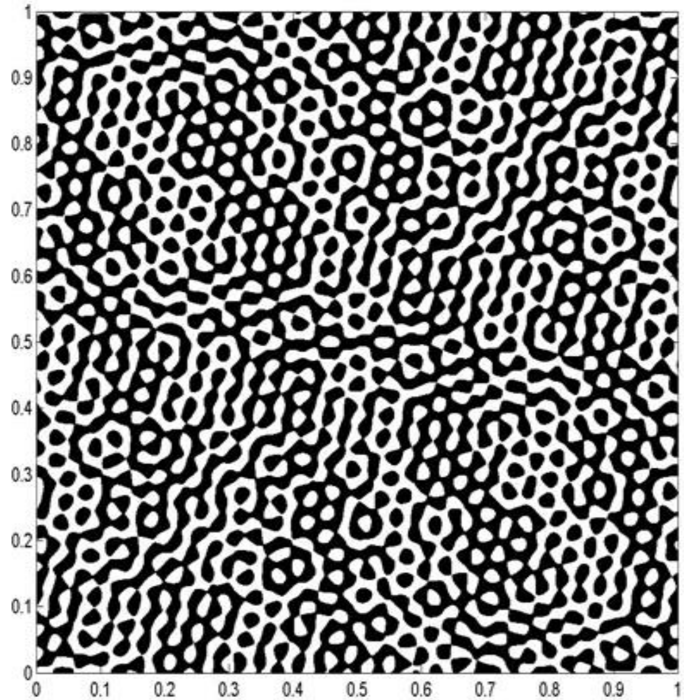
Exceptional "Cilleruelo" $r_2(\mathrm{n}) \to \infty$

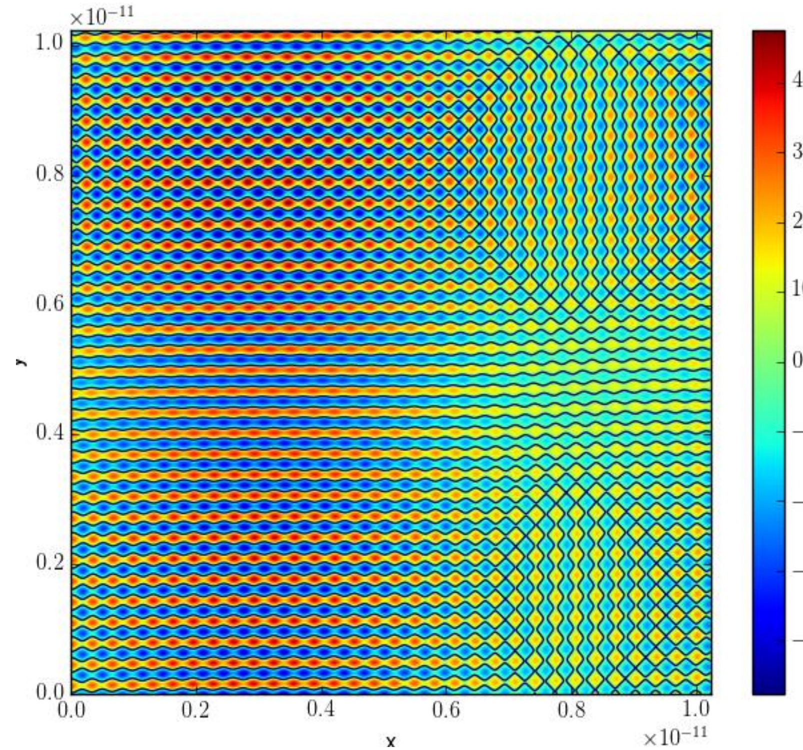




- Partial classification (P. Kurlberg-IW `15)

Some pics





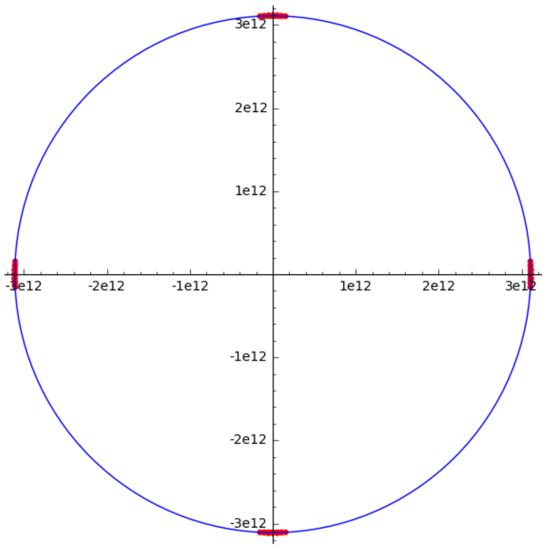
$n = 1105$

32 directions

$n = 967641808851334762447465 3$

256 directions

Fragment, domains long and narrow



On the 2 squares problem

- $\tau_n = \frac{1}{r_2(n)} \sum_{\|\mu\|^2=n} \delta_{\mu/\sqrt{n}}$ probability $\mathcal{S}^1$ (spectral measure)
- Equidistributed $\tau_{n_j} \Rightarrow \frac{d\theta}{2\pi}$ $(\tau_{n_j} \Rightarrow \tau)$
- Angular distribution $\leftrightsquigarrow$ Nodal structure

Local: Krishnapur-Kurlberg-W `13, Rudnick-W `14, Rossi-W `17

- Nonlocal: $\mathrm{N}(f_n(x))$ – total nodal count

Nazarov-Sodin `12,`15+Kurlberg-W `14,17

Toral eigenfunctions

- $f_n(x) = \sum_{\|\mu\|^2=n} a_\mu \cdot e(\langle x, \mu \rangle)$ arithmetic random waves
- $\tau_n = \frac{1}{r_2(n)} \sum_{\|\mu\|^2=n} \delta_{\mu/\sqrt{n}}$ on S^1
- Apply N-S: if $\tau_n \Rightarrow \tau$ then $c = c_{NS}(\tau)$ (generalised)

$$E[N(f_n(x))] = c \cdot n + o(n)$$

- Generic $E\left[\frac{|N(f_n(x)) - c_{RWM}|}{n}\right] \to 0$
- Exponential concentration (Y. Rozenshein `15)

Toral eigenfunctions (cont.)

- $f_n(x) = \sum_{\|\mu\|^2=n} a_\mu \cdot e(\langle x, \mu\rangle)$ arithmetic random waves
- $\tau_n = \frac{1}{r_2(n)} \sum_{\|\mu\|^2=n} \delta_{\mu/\sqrt{n}}$ on S^1
- Theorem 2 (Kurlberg-W):

1. Uniformly

$$E[N(f_n(x))] = c_{NS}(\tau_n) \cdot n + O(\sqrt{n})$$

2. $c_{NS}(\tau) = 0$ iff τ is Cilleruelo or its tilt

(τ restricted, in particular by symmetries)

Toral eigenfunctions (cont.)

- $c_{NS}(\tau) = 0$ iff τ is Cilleruelo or its tilt (restricted)
- $c_{NS}(\tau)$ attains an interval $[0, c_1]$.
- Q.: Is it true that $c_1 = c_0 = c_{RWM}$ uniquely?
- Q.: For Cilleruelo: $E[N(f_n(x))]$ - ?

  $E[N(f_n(x))] \to \infty$ - ?

  $E[N(f_n(x))] \gg \sqrt{n}$ - ?
- Meanwhile **full** classification $c_{NS}(\rho) = 0$ (Beliaev-McAuley-Muirhead). Same for $d_{NS}(\rho)$?