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By Dmitri Beliaev and Stanislav Smirnov

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Abstract

In many problems of classical analysis extremal configurations appear to exhibit complicated fractal structures, making it hard to describe them and to attack such problems. This is particularly true for questions related to the *multifractal analysis* of *harmonic measure*. We argue that, searching for extremals in such problems, one should work with random fractals rather than deterministic ones. We introduce a new class of fractals: *random conformal snowflakes*, and investigate their properties, developing tools to estimate spectra and showing that extremals can be found in this class. As an application we significantly improve known estimates from below on the extremal behavior of harmonic measure, showing how to construct a rather simple snowflake, which has a spectrum quite close to the conjectured extremal value.

1. Introduction

It became apparent during the last decade that extremal configurations in many important problems in classical complex analysis exhibit complicated fractal structures. This makes such problems more difficult to approach than similar ones, where extremal objects are smooth.

As an example one can consider the coefficient problem for univalent functions. Bieberbach formulated his famous conjecture arguing that the Köebe function, which maps a unit disc to a plane with a straight slit, is extremal. The Bieberbach conjecture was ultimately proved by de Branges in 1985 [5], while the sharp growth asymptotics was obtained by Littlewood [8] in 1925 by a much easier argument.

Yet, the coefficient growth problem for bounded functions remains widely open, largely due to the fact that the extremals must be of fractal nature (cf. [4]). This relates (see [1], [3]) to a more general problem of finding the *universal*

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multifractal spectrum of *harmonic measure* defined below, which includes many other problems, in particular conjectures of Brennan, Carleson and Jones, Kraetzer, Szegö, and Littlewood.

In this paper we report on our search for extremal fractals. We argue that one should study random fractals instead of deterministic ones. We introduce a new class of random fractals, *random conformal snowflakes*, investigate their properties, and as a consequence significantly improve known estimates from below for the multifractal spectra of harmonic measure.

1.1. *Multifractal analysis of harmonic measure.* Recently, it became clear that appropriate language for many problems in geometric function theory is given by the *multifractal analysis* of *harmonic measure*.

Harmonic measure plays an important role in geometric function theory and many other areas of mathematics. There are several equivalent definitions; in the two-dimensional, simply connected case it could be described as the normalized image of Lebesgue measure under the Riemann mapping.

The concept of multifractal spectrum of a measure was introduced by Mandelbrot in 1971 in [11], [12] in papers devoted to the distribution of energy in a turbulent flow. We use the definitions that appeared in 1986 in a seminal physics paper [6] by Halsey, Jensen, Kadanoff, Procaccia, Shraiman who tried to understand and describe scaling laws of physical measures on different fractals of physical nature (strange attractors, stochastic fractals like DLA, etc.).

There are various notions of spectra and several ways to make a rigorous definition. Two standard notions are *packing* and *dimension* spectra. The packing spectrum of a measure ω is defined as

$$\pi_\Omega(t) = \sup \left\{q :\ \forall\delta > 0\ \exists\ \delta-\text{packing}\ \{B\}\ \text{with}\ \sum \operatorname{diam}(B)^t \omega(B)^q\ \geq\ 1\right\},$$

where δ-packing is a collection of disjoint open sets whose diameters do not exceed δ.

The *dimension spectrum* of harmonic measure gives the dimension of the set of points, where harmonic measure satisfies a certain power law:

$$f(\alpha)\ :=\ \dim\left\{z :\ \omega\left(B(z,\delta)\right)\ \approx\ \delta^\alpha\ ,\ \delta \to 0\right\},\ \alpha \geq \tfrac{1}{2}\ .$$

Here dim stands for the Hausdorff or Minkowski dimension, leading to possibly different spectra. The restriction $\alpha \geq 1/2$ is specific for harmonic measure on planar simply-connected domains and is due to Beurling's inequality. Of course, in general there will be many points where measure behaves differently at different scales, so one has to add lim sups and lim infs to the definition above; consult [9] for details.

In our context it is more suitable to work with a modification of the packing spectrum which is specific for the harmonic measure on a two-dimensional, simply connected domain Ω. In this case we can define the *integral means spectrum* as

$$\beta_\phi(t) \ := \ \limsup_{r\to 1+} \frac{\log \int_0^{2\pi} |\phi'(re^{i\theta})|^t d\theta}{|\log(r-1)|}, \ t \in \mathbb{R} \ ,$$

where ϕ is a Riemann map from the complement of the unit disc onto a simply connected domain Ω.

Connections between all these spectra for particular domains are not necessarily simple, but the *universal spectra*

$$\Pi(t) = \sup_\Omega \pi(t), \quad F(\alpha) = \sup_\Omega f(\alpha), \ \text{ and } \ B(t) = \sup_\Omega \beta(t),$$

(where the suprema are taken over all simply-connected planar domains Ω) are related by Legendre-type transforms:

$$\begin{aligned} F(\alpha) &= \inf_{0\le t\le 2} (\alpha \Pi(t) + t), \quad \alpha \ge 1 \ , \\ \Pi(t) &= \sup_{\alpha\ge 1} \left(\frac{F(\alpha) - t)}{\alpha} \right), \quad 0 \le t \le 2 \ , \\ \Pi(t) &= B(t) - t + 1 \ . \end{aligned}$$

See Makarov's survey [9] for details.

1.2. *Random fractals.* One of the main problems in the computation of the integral means spectrum (or other multifractal spectra) is the fact that the derivative of a Riemann map for a fractal domain depends on the argument in a very nonregular way: ϕ' is a "fractal" object in itself. We propose to study random fractals to overcome this problem. For a random function ϕ it is natural to consider the *average integral means spectrum*:

$$\begin{aligned} \overline{\beta}(t) &= \sup \left\{ \beta : \int_1 (r-1)^{\beta-1} \int_0^{2\pi} \mathbb{E}\left[|f'(re^{i\theta})|^t \right] d\theta dr = \infty \right\} \\ &= \inf \left\{ \beta : \int_1 (r-1)^{\beta-1} \int_0^{2\pi} \mathbb{E}\left[|f'(re^{i\theta})|^t \right] d\theta dr < \infty \right\} . \end{aligned}$$

The average spectrum does not have to be related to the spectra of a particular realization. We want to point out that even if ϕ has the same spectrum a.s. it does not guarantee that $\overline{\beta}(t)$ is equal to the a.s. value of $\beta(t)$. Moreover, it can happen that $\overline{\beta}$ is not a spectrum of *any* particular domain.

On the other hand, one can see that $\overline{\beta}(t)$ is bounded by the universal spectrum $B(t)$. Indeed, suppose that there is a random f with $\overline{\beta}(t) > B + \varepsilon$, then for small r there are particular realizations of f with $\int |f'(z)| d\theta > (r-1)^{-B-\varepsilon/2}$. Then by Makarov's fractal approximation [9] there is a (deterministic) function F such that $\beta_F(t) > B(t)$ which is impossible by the definition of $B(t)$.

For many classes of random fractals, $\mathbb{E}|\phi'|^t$ (or its growth rate) does not depend on the argument. This allows us to drop the integration with respect to the argument and study the growth rate along any particular radius. Perhaps more importantly, $\mathbb{E}|\phi'|$ is no longer a "fractal" function.

One can think that this is not a big advantage compared to the usual integral means spectrum: instead of averaging over different arguments we average over different realizations of a fractal. But most fractals are results of some kind of an iterative construction, which means that they are invariant under some (random) transformation. Thus $\mathbb{E}|\phi'|^t$ is a solution of a certain equation. Solving this equation (or estimating its solutions) we can find $\bar{\beta}(t)$.

In this paper we want to show how one can employ these ideas. In Section 2 we introduce a new class of random fractals that we call random conformal snowflakes. In Section 3 we show that $\bar{\beta}(t)$ for this class is related to the main eigenvalue of a particular integral operator. We also prove the fractal approximation for this class in Section 4. In Appendix 5 we give an example of a snowflake and prove that for this snowflake $\bar{\beta}(1) > 0.23$. This significantly improves the previously known estimate $B(1) > 0.17$ due to Pommerenke [14].

2. **Conformal snowflake**

The construction of our conformal snowflake is similar to the construction in Pommerenke's paper [13]. The main difference is the introduction of randomness.

By Σ' we denote a class of all univalent functions $\phi : \mathbb{D}_- \to \mathbb{D}_-$ such that $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$. Let $\phi \in \Sigma'$ be a function with expansion at infinity $\phi(z) = b_1 z + \dots$. By $\operatorname{cap}\phi = \operatorname{cap}\Omega$ we denote the logarithmic capacity of ϕ which is equal to $\log|b_1|$. We will also use the so called *Koebe n-root transform* which is defined as

$$(K\phi)(z) = (K_n\phi)(z) = \sqrt[n]{\phi(z^n)}.$$

It is a well known fact that the Koebe transform is well defined and $K\phi \in \Sigma'$. It is easy to check that the Koebe transform divides capacity by n and that the capacity of a composition is the sum of capacities.

First, we define the deterministic snowflake. To construct a snowflake we need a building block $\phi \in \Sigma'$ and an integer $k \geq 2$. Our snowflake will be the result of the following iterative procedure: we start with the building block and at the n-th step take a composition of our function and the k^n-root transform of the rotated building block.

Notation 1. Let $\phi \in \Sigma'$ and $\theta \in [0, 2\pi]$. By $\phi_\theta(z)$ we denote the map from Σ' whose range is the rotation of that for ϕ, namely $e^{i\theta}\phi(ze^{-i\theta})$.

Definition 1. Let $\phi \in \Sigma'$, $k \geq 2$ be an integer, and $\{\theta_n\}$ be a sequence of numbers from $\mathbb{T}$. Let $f_0(z) = \phi_{\theta_0}(z)$ and

$$f_n(z) = f_{n-1}(K_{k^n}\phi_{\theta_n}(z)) = \phi_{\theta_0}(\phi_{\theta_1}^{1/k}(\dots\phi_{\theta_n}^{1/k}(z^{k^n})\dots).$$

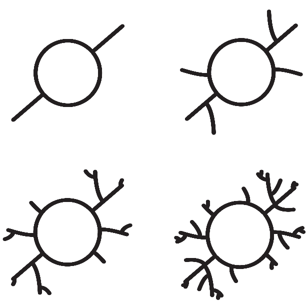

Figure 1. Images of $\bar{f}_1$, $\bar{f}_2$, $\bar{f}_3$, and $\bar{f}_4$

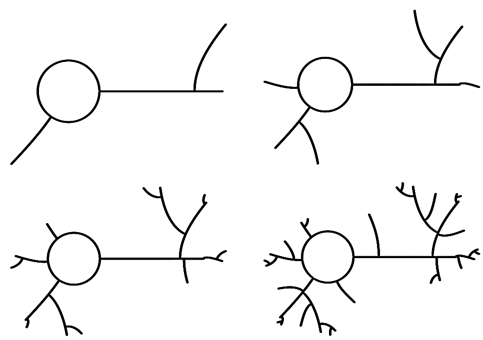

Figure 2. Images of f_1, f_2, f_3, and f_4

The *conformal snowflake* f is the limit of f_n. For simplicity $S = \mathbb{C} \setminus f(\mathbb{D}_-)$ and $g = f^{-1}$ are also called snowflakes.

Sometimes it is easier to work with a slightly different symmetric snowflake

$$\bar{f}_n(z) = \phi_{\theta_1}^{1/k}(\ldots \phi_{\theta_n}^{1/k}(z^{k^n})\ldots) = \Phi_1 \circ \cdots \circ \Phi_n(z),$$

where $\Phi_j = K_{k^j}\phi_{\theta_j}$. There are two equivalent ways to construct the symmetric snowflake from the usual one. One is to take the Koebe transform Kf_n, another is to start with $f_0(z) = z$. It is easy to see that $f_n = \Phi_0 \circ \cdots \circ \Phi_n$.

How does this snowflake grow? This is easy to analyze if we look at the evolution of $\bar{f}_n$. At every step we add k^n equidistributed (according to the harmonic measure) small copies of the building block. But they are not exact copies; they are distorted a little bit by a conformal mapping.

Figures 1 and 2 show images of the first four functions $\bar{f}$ and f with $k = 2$ and the building block is a slit map (which adds a straight slit of length 4).

LEMMA 2.1. *Let* $f_n = \phi_{\theta_0}(\bar{f}_n(z))$ *be the* n*-th approximation to the snowflake with a building block* ϕ *and* $k \geq 2$. *Then* cap (f_n) *and* cap $(\bar{f}_n)$ *are bounded by* (*and converge to*) cap $(\phi)k/(k-1)$ *and* cap $(\phi)/(k-1)$.

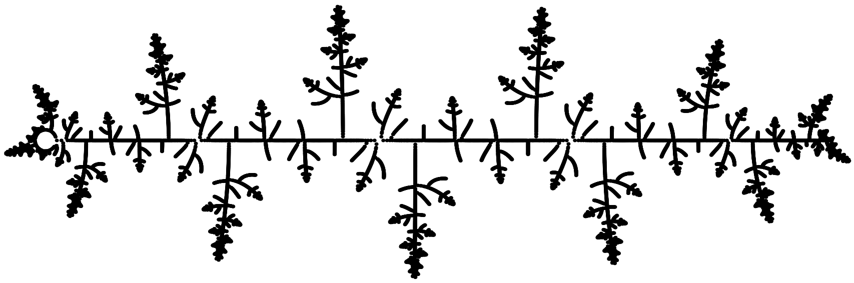

Figure 3. The third generation of a snowflake: f_3.

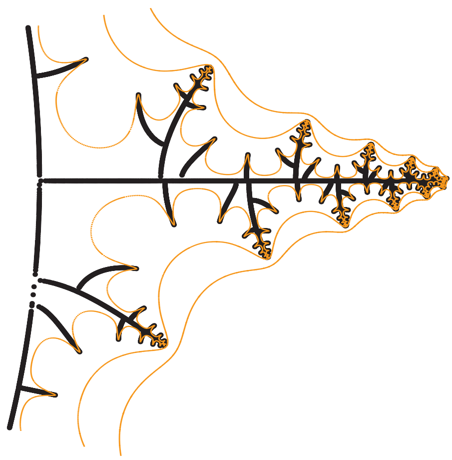

Figure 4. The image of a small boundary arc under $\bar{f}_3$ with three Green's lines.

Proof. This lemma follows immediately from the standard facts that

$$\operatorname{cap}(f \circ g) = \operatorname{cap}(f) + \operatorname{cap}(g),$$
$$\operatorname{cap}(K_n f) = \operatorname{cap}(f)/n. \qquad \square$$

THEOREM 2.2. *The conformal snowflake is well defined, namely let f_n be an n-th approximation to a snowflake with a building block ϕ and $k \geq 2$. Then there is $f \in \Sigma'$ such that the f_n converge to f uniformly on every compact subset of $\mathbb{D}_-$.*

Proof. Fix $\varepsilon > 0$. It is enough to prove that $\bar{f}_n$ converge uniformly on $\mathbb{D}_\varepsilon = \{z : |z| \geq 1 + \varepsilon\}$. Suppose that $m > n$ so that we can write $\bar{f}_m = \bar{f}_n \circ \Phi_{n,m}$ where $\Phi_{n,m} = \Phi_{n+1} \circ \cdots \circ \Phi_m$ and

$$|\bar{f}_n(z) - \bar{f}_m(z)| = |\bar{f}_n(z) - \bar{f}_n(\Phi_{n,m}(z))| \leq \max_{\zeta \in \mathbb{D}_\varepsilon} |\bar{f}_n'(\zeta)||z - \Phi_{n,m}(z)|.$$

By Lemma 2.1 cap $(\bar{f_n})$ is uniformly bounded; hence by the Koebe distortion theorem (Theorem 1.6 in [14]) the derivative of $\bar{f_n}$ is uniformly bounded in $\mathbb{D}_\varepsilon$. Thus it is enough to prove that $\Phi_{n,m}(z)$ converge uniformly to z.

Let $\phi(z) = b_1 z + \dots$ at infinity, then $\Phi_n(z) = b_1^{1/k^n} z + \dots$; functions $\Phi_{n,m}$ have expansion

$$b_1^{k^{-n}+\dots+k^{-m}} z + \dots = b_1^{(n,m)} z + \dots .$$

Obviously, $b_1^{(n,m)} \to 1$ as $n \to \infty$. This proves that $\Phi_{n,m}(z) \to z$ uniformly on $\mathbb{D}_\varepsilon$; hence, f_n converge uniformly. Uniform limits of functions from Σ' can be either a constant or a function from Σ'. Since cap (f_n) is uniformly bounded the limit cannot be a constant. □

Definition 2. Let $\phi \in \Sigma'$ and $k \geq 2$ be an integer. The *random conformal snowflake* is a conformal snowflake defined by ϕ, k, and $\{\theta_n\}$, where θ_n are independent random variables uniformly distributed on $\mathbb{T}$.

THEOREM 2.3. *Let $\phi \in \Sigma'$, $k \geq 2$ be an integer, and $\psi = \phi^{-1}$. Let f be a corresponding random snowflake and $g = f^{-1}$. Then the distribution of f is invariant under the transformation $\Sigma' \times \mathbb{T} \mapsto \Sigma'$ which is defined by*

$$(f, \theta) \mapsto \phi_\theta(K_k f).$$

In other words

$$f(z) = \phi_\theta((K_k f)(z)) = \phi_\theta(f^{1/k}(z^k)), \tag{1}$$

$$g(z) = (K_k g)(\psi_\theta(z)) = g^{1/k}(\psi_\theta^k(z)), \tag{2}$$

where θ is uniformly distributed on $\mathbb{T}$. Both equalities should be understood in the sense of distributions, i.e. distributions of both parts are the same.

Proof. Let f be a snowflake generated by $\{\theta_n\}$. The probability distribution of the family of snowflakes is the infinite product of (normalized) Lebesgue measures on $\mathbb{T}$. By the definition

$$f(z) = \lim_{n\to\infty} \phi_{\theta_0}(\phi_{\theta_1}^{1/k}(\dots \phi_{\theta_n}^{1/k}(z^{k^n})\dots)$$

and

$$\phi_\theta((K_k f)(z)) = \lim_{n\to\infty} \phi_\theta(\phi_{\theta_0}^{1/k}(\dots \phi_{\theta_n}^{1/k}(z^{k^{n+1}})\dots).$$

Hence $\phi_\theta(K_k f)$ is just the snowflake defined by the sequence $\theta, \theta_0, \theta_1, \dots$. So the transformation $f(z) \mapsto \phi_\theta((K_k f)(z))$ is just a shift on the $[-\pi, \pi]^{\mathbb{N}}$. Obviously the product measure is invariant under the shift. This proves stationarity of f. Stationarity of g follows immediately from stationarity of f. □

There is another way to think about random snowflakes. Let $\mathcal{M}$ be a space of probability measures on Σ'. And let T be a random transformation $f \mapsto \phi_\theta(K_k f)$, where θ is uniformly distributed on $[\pi, \pi]$. Obviously T acts on $\mathcal{M}$. The distribution of a random snowflake is the only measure which is invariant under T. In some sense the random snowflake is an analog of a Julia set: it semi-conjugates z^k and ψ_θ^k (see Figure 5).

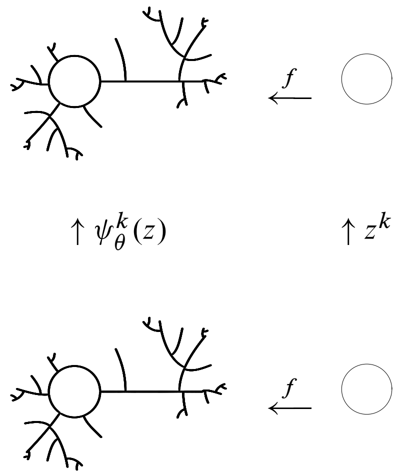

Figure 5. Conformal snowflake f semi-conjugates z^k and ψ_θ^k.

The random conformal snowflakes are also rotationally invariant; the exact meaning is given by the following theorem.

THEOREM 2.4. *Let $\phi \in \Sigma'$, $k \geq 2$, and g be the corresponding snowflake. Then g is rotationally invariant; namely $g(z)$ and $e^{i\omega}g(e^{-i\omega}z)$ have the same distribution for any ω.*

Proof. Let $g_n(z)$ be the nth approximation to the snowflake defined by the sequence of rotations $\theta_0, \dots, \theta_n$. We claim that $\tilde{g}(z) = e^{i\omega}g(e^{-i\omega}z)$ is the approximation to the snowflake defined by $\tilde{\theta}_0, \dots, \tilde{\theta}_n$ where $\tilde{\theta}_j = \theta_j + \omega k^j$ (we add arguments mod 2π).

We prove this by induction. Obviously this is true for $\tilde{g}_0$. Suppose that it is true for $\tilde{g}_{n-1}$. By the definition of g_n and the assumption that $g_{n-1}(e^{-i\omega}z) = e^{-i\omega}\tilde{g}_{n-1}(z)$ we have that

$$\begin{aligned} e^{i\omega}g_n(e^{-i\omega}z) &= e^{i\omega}e^{i\theta_n/k^n}\psi^{1/k^n}(e^{-i\theta_n}g_{n-1}^{k^n}(e^{-i\omega}z)) \\ &= e^{i\tilde{\theta}_n/k^n}\psi^{1/k^n}(e^{i\tilde{\theta}_n}\tilde{g}_{n-1}^{k^n}(z)) = \tilde{g}_n(z). \end{aligned}$$

Obviously $\tilde{\theta}_n$ are also independent and uniformly distributed on $\mathbb{T}$; hence $\tilde{g}_n$ has the same distribution as g_n. □

COROLLARY 2.5. *The distributions of $|g(z)|$ and $|g'(z)|$ depend on $|z|$ only. The same is true for f.*

3. Spectrum of a conformal snowflake

As discussed above, for random fractals it is more natural to consider the average spectrum $\bar{\beta}(t)$ instead of the usual spectrum $\beta(t)$. We will work with $\bar{\beta}(t)$ only and "spectrum" will always mean $\bar{\beta}(t)$.

Notation 2. We will write $\mathcal{L}$ for the class of functions on $(1,\infty)$ that are bounded on compact sets and integrable in the neighborhood of 1. In particular, these functions belong to $L^1[1,R]$ for any $1<R<\infty$.

Let $F(z)=F(|z|)=F(r)=\mathbb{E}\,[\,|g'(r)/g(r)|^\tau \log^\sigma|g(r)|]$ where $\tau=2-t$ and $\sigma=\beta-1$.

LEMMA 3.1. *The $\overline{\beta}(t)$ spectrum of the snowflake is equal to*

$$\inf\{\beta : F(r)\in\mathcal{L}\}\,.$$

Proof. By the definition $\overline{\beta}$ is the minimal value of β such that

$$\int_1\int_0^{2\pi}(r-1)^{\beta-1}\mathbb{E}|f(re^{i\theta})|^t\,d\theta dr$$

is finite. We change the variable to $w=f(z)=f(re^{i\theta})$,

$$\begin{aligned}\int\int\mathbb{E}\left[|f'(re^{i\theta})|^t\right](r-1)^{\beta-1}drd\theta &= \int\frac{\mathbb{E}\left[|f'(z)|^t\right](|z|-1)^{\beta-1}dm(z)}{r}\\ &= \int\mathbb{E}\left[\frac{|g'(w)|^{2-t}(|g(w)|-1)^{\beta-1}}{|g(w)|}\right]dm\,,\end{aligned}$$

where m is the Lebesgue measure. Note that $|g|$ is uniformly bounded and g is rotationally invariant; hence the last integral is finite if and only if

$$\int_1|g'(r)|^\tau(|g(r)|-1)^\sigma dr<\infty.$$

Since $1<|g|$ is uniformly bounded we have that $|g'(r)|^\tau(|g(r)|-1)^\sigma$ is comparable up to an absolute constant to

$$\left(\frac{|g'(r)|}{|g(r)|}\right)^\tau\log^\sigma|g(r)|. \qquad \square$$

LEMMA 3.2. *If $F\in\mathcal{L}$ then it is a solution of the following equation*:

$$F(r)=\frac{1}{k^\sigma}\int_{-\pi}^{\pi}F(|\psi^k(re^{i\theta})|)|\psi^{k-1}(re^{i\theta})\psi'(re^{i\theta})|^\tau\frac{d\theta}{2\pi}. \tag{3}$$

Proof. By Theorem 2.3 $g(z)$ and $g^{1/k}(\psi_\theta^k(z))$ have the same distribution; hence

$$\begin{aligned}F(r)&=\mathbb{E}\left[\left|g'(r)/g(r)\right|^\tau\log^\sigma|g(r)|\right]\\ &=\mathbb{E}\left[\left|\frac{(g^{1/k}(\psi_\theta^k(r)))'}{g^{1/k}(\psi_\theta^k(r))}\right|^\tau\log^\sigma|g^{1/k}(\psi_\theta^k(r))|\right]\\ &=\mathbb{E}\left[\left|\frac{g'(\psi_\theta^k(r))}{g(\psi_\theta^k(r))}\right|^\tau\log^\sigma|g(\psi_\theta^k(r))|\frac{|\psi_\theta'(r)\psi_\theta^{k-1}(r)|^\tau}{k^\sigma}\right],\end{aligned}$$

where θ has a uniform distribution. The expectation is the integral with respect to the joint distribution of g and θ; since they are independent this joint distribution is just a product measure. So we can write it as the double integral: first, we take the expectation with respect to the distribution of g and then with respect to the (uniform) distribution of θ:

$$\begin{aligned} F(r) &= \int_{-\pi}^{\pi} \left(\int \left| \frac{g'(\psi_\theta^k(r))}{g(\psi_\theta^k(r))} \right|^\tau \log^\sigma |g(\psi_\theta^k(r))| \frac{|\psi_\theta'(r)\psi_\theta^{k-1}(r)|^\tau}{k^\sigma} dg \right) \frac{d\theta}{2\pi} \\ &= \int_{-\pi}^{\pi} \left(\int \left| \frac{g'(\psi_\theta^k(r))}{g(\psi_\theta^k(r))} \right|^\tau \log^\sigma |g(\psi_\theta^k(r))| dg \right) \frac{|\psi_\theta'(r)\psi_\theta^{k-1}(r)|^\tau}{k^\sigma} \frac{d\theta}{2\pi}. \end{aligned}$$

The inner integral is equal to $F(\psi_\theta^k(r)) = F(\psi^k(e^{-i\theta}r))$ by the definition of F; hence

$$\begin{aligned} F(r) &= \int_{-\pi}^{\pi} F(\psi^k(e^{-i\theta}r)) \frac{|\psi'(e^{-i\theta}r)\psi^{k-1}(e^{-i\theta}r)|^\tau}{k^\sigma} \frac{d\theta}{2\pi} \\ &= \frac{1}{k^\sigma} \int_{-\pi}^{\pi} F(\psi^k(e^{i\theta}r)) |\psi'(e^{i\theta}r)\psi^{k-1}(e^{i\theta}r)|^\tau \frac{d\theta}{2\pi} \end{aligned}$$

which completes the proof. □

This equation is the key ingredient in our calculations. One thinks of F as the main eigenfunction of an integral operator. Hence the problem of finding the spectrum of the snowflake boils down to the question about the main eigenvalue of a particular integral operator. Usually it is not very difficult to estimate the latter.

This justifies the definition:

$$(Qf)(r) := k \int_{-\pi}^{\pi} f(|\psi^k(re^{i\theta})|) \, |\psi^{k-1}(re^{i\theta})\psi'(re^{i\theta})|^\tau \frac{d\theta}{2\pi} \, . \tag{4}$$

Using this notation we can rewrite (3) as

$$k^\beta F = QF.$$

Note that this is in fact an ordinary kernel operator and $|\psi|$ is a smooth function of θ. Hence we can change the variable and write it as an integral operator. As mentioned above, the study of a F is closely related to the study of operator Q and its eigenvalues. Our estimate of the spectrum is in fact an estimate of the main eigenvalue.

3.1. *Adjoint operator.* First of all we want to find a formally adjoint operator. Let ν be a bounded test function and $R > 1$ such that $D_R \subset \psi^k(D_R)$ where $D_R = \{z : 1 < |z| < R\}$. We have

$$\int_1^R Qf(r)\nu(r)dr = \int_1^R \nu(r)k\int_0^{2\pi} f(\psi^k(re^{i\theta}))\,|\psi'(re^{i\theta})\psi^{k-1}(re^{i\theta})|^\tau \frac{d\theta}{2\pi}dr$$
$$= \int_{D_R} \frac{\nu(|z|)}{|z|}\frac{k}{2\pi} f(\psi^k(z))|\psi'(z)\psi^{k-1}(z)|^\tau dm(z),$$

where dm is the Lebesgue measure. Changing a variable to $w = \psi^k(z)$ we get

$$\int_{\psi^k(D_R)} \frac{\nu(\phi(w^{1/k}))}{|\phi(w^{1/k})|}\frac{1}{k2\pi} f(w)|\phi'(w^{1/k})w^{1/k-1}|^{2-\tau}dm(w)$$
$$\geq \int_1^R\int_0^{2\pi k} \frac{r\nu(\phi(r^{1/k}e^{i\theta/k}))}{|\phi(r^{1/k}e^{i\theta/k})|} f(r)r^{\frac{(1-k)(\tau-2)}{k}}|\phi'(r^{1/k}e^{i\theta/k})|^{2-\tau}\frac{d\theta}{2\pi k}dr$$
$$= \int_1^R f(r)\int_0^{2\pi} \frac{r\nu(\phi(r^{1/k}e^{i\theta}))}{|\phi(r^{1/k}e^{i\theta})|} r^{\frac{(1-k)(\tau-2)}{k}}|\phi'(r^{1/k}e^{i\theta})|^{2-\tau}\frac{d\theta}{2\pi}dr.$$

So we define another operator

$$P\nu(r) := r^{1-\frac{(k-1)(2-\tau)}{k}}\int_0^{2\pi}\frac{\nu(\phi(r^{1/k}e^{i\theta}))}{|\phi(r^{1/k}e^{i\theta})|}|\phi'(r^{1/k}e^{i\theta})|^{2-\tau}\frac{d\theta}{2\pi}. \tag{5}$$

Changing $2-\tau$ to t we can rewrite (5) as

$$P\nu(r) := r^{1-\frac{(k-1)t}{k}}\int_0^{2\pi}\frac{\nu(\phi(r^{1/k}e^{i\theta}))}{|\phi(r^{1/k}e^{i\theta})|}|\phi'(r^{1/k}e^{i\theta})|^{t}\frac{d\theta}{2\pi}. \tag{6}$$

The inequality above can be written as

$$\int_1^R Qf(r)\nu(r)dr \geq \int_1^R f(r)P\nu(r)dr. \tag{7}$$

We would like to note that for $R = \infty$ there is an equality since $\psi^K(\mathbb{D}_-)$ covers $\mathbb{D}_-$ exactly k times. In this case

$$\int_1^\infty Qf(r)\nu(r)dr = \int_1^\infty f(r)P\nu(r)dr. \tag{8}$$

so that operators P and Q are formally adjoint on $[1,\infty)$.

LEMMA 3.3. *Operator $Q = Q(t)$ acts on $\mathcal{L}$ if $\int |\phi'(re^{i\theta})|d\theta$ is bounded. If $t \geq 1$ then Q also acts on $L^1(1,\infty)$.*

Proof. Let $\nu = 1$ in (8). Then

$$\int_1^\infty Qf(r)dr = \int_1^\infty f(r)\int_{-\pi}^{\pi}\frac{r^{1-\frac{(k-1)t}{k}}}{|\phi(r^{1/k}e^{i\theta})|}|\phi'(r^{1/k}e^{i\theta})|^t\frac{d\theta}{2\pi}.$$

Let $r < R$, then

$$\frac{r^{1-\frac{(k-1)t}{k}}}{|\phi(r^{1/k}e^{i\theta})|} < R^{1-\frac{(k-1)t}{k}},$$

so that the second integral is bounded since $\int |\phi'|^t d\theta$ is bounded. This proves that Qf is in $\mathcal{L}$. To prove that it acts on $L^1(1,\infty)$ we should consider the large values of r. At infinity $\phi(z) = cz + \ldots$ and $\phi'(z) = c + \ldots$. Hence

$$\frac{r^{1-\frac{(k-1)t}{k}}}{|\phi(r^{1/k}e^{i\theta})|}|\phi'(r^{1/k}e^{i\theta})|^t \approx \frac{r^{1-\frac{(k-1)t}{k}}|c|^t}{|c|r^{1/k}} = \operatorname{const} r^{1-\frac{(k-1)t)}{k}-\frac{1}{k}}$$
$$= \operatorname{const} r^{\frac{(k-1)(1-t)}{k}}.$$

Thus the second integral is comparable (up to a universal constant) to $r^{\frac{(k-1)(1-t)}{k}}$, and so it is bounded if and only if $t \geq 1$. □

Remark 1. Note that the assumption on the integral of $|\phi'|$ is just a bit stronger than $\beta_\phi(t) = 0$. We restrict ourselves to the building blocks that are smooth up to the boundary. For such building blocks this assumption is always true. Condition $t \geq 1$ is technical and due to the behavior at infinity which should be irrelevant. Introducing the weight at infinity we can get rid of this assumption.

Next we want to discuss how eigenvalues of P and Q are related to the spectrum of the snowflake. If F is integrable, then it is a solution of (3) and using (7) we can write

$$\int_1^R F(r)\nu(r) = \int_1^R \frac{QF(r)}{k^{1+\sigma}}\nu(r) \geq \int_1^R F(r)\nu(r)\frac{P\nu(r)}{\nu(r)k^{\sigma+1}}. \tag{9}$$

Suppose that t is fixed. Now, fix a positive test function ν. If $P\nu(r) \geq \nu(r)k^{\sigma+1}$ then we arrive at a contradiction. This means that $F(r)$ for this particular pair of τ and σ cannot be integrable. Using this fact we can estimate $\overline{\beta}(t)$ from below. Hence any positive ν gives the lower bound of the spectrum.

$$\overline{\beta}(t) \geq \min_{1\leq r\leq R} \log\left(\frac{P\nu(r)}{\nu(r)}\right) / \log k. \tag{10}$$

Obviously, the best choice of ν is an eigenfunction of P corresponding to the maximal eigenvalue. This proves the following lemma:

LEMMA 3.4. *Let λ be the maximal eigenvalue of P (on any interval $[1, R]$ such that $D_R \subset \psi^k(D_R)$), then $\overline{\beta}(t) \geq \log\lambda/\log k$.*

4. **Fractal approximation**

In this section we establish the fractal approximation by conformal snowflakes. Namely we show that for any t one can construct a snowflake with building block which is smooth up to the boundary and with $\overline{\beta}(t)$ arbitrarily close to $B(t)$. The proof of this theorem is similar to the proof of the fractal approximation for standard snowflakes but it is less technical.

THEOREM 4.1. *For any ε and t there are a building block*

$$\phi \in \Sigma' \cap C^\infty(\{|z| \geq 1\})$$

and a positive integer k that define the snowflake with $\overline{\beta}(t) > B(t) - \varepsilon$.

We will use the following lemma.

LEMMA 4.2. *For any* $\varepsilon > 0$, $t \in \mathbb{R}$, *there is* $A > 0$ *such that for any* $\delta > 0$ *there is a function* $\phi \in \Sigma' \cap C^\infty$ *such that*

$$\int \left|\phi'(re^{i\theta})\right|^t d\theta > A\left(\frac{1}{\delta}\right)^{B(t)-\varepsilon}$$

for $\delta > r-1$. *Moreover, the capacity of* ϕ *is bounded by a universal constant that does not depend on* δ.

Proof. There is a function f with $\beta_f(t) > B(t) - \varepsilon$. Hence there is a constant A such that

$$\int \left|f'(re^{i\theta})\right|^t d\theta > A(r-1)^{-B(t)+2\varepsilon}.$$

The only problem is that this function is not smooth up to the boundary. Set $\phi(z) = f(sz)$. Obviously, $\phi(z) \rightrightarrows f(z)$ as $s \to 1$. If we fix a scale δ then there is s sufficiently close to 1 such that $\int |\phi'/\phi|^t d\theta > A\delta^{-B(t)+2\varepsilon}/2$. But for $r < 1+\delta$ the integral cannot be smaller by the subharmonicity. □

Proof of Theorem 4.1. It is easy to see that

$$\operatorname{cap}(f_n) < \operatorname{cap}(f) = \operatorname{cap}(\phi)/(1-1/k) < 2\operatorname{cap}(\phi),$$

hence $\operatorname{cap}(f_n)$ and $|f_n(z)|$ for $|z| < 2$ are bounded by the universal constants that depend on capacity of ϕ only and do not depend on k. It also follows that $|K_k f_n(z)| < 1 + c/k$ for $|z| < 2$ and c depending on $\operatorname{cap}(\phi)$ only.

Let us fix t and let ϕ be a function from Lemma 4.2 for $\delta = c/k$. By $I(f, \delta)$ we denote

$$\int_{-\pi}^{\pi} \left|f'(re^{i\theta})\right|^t d\theta,$$

where $r = \exp(\delta)$.

The k-root transform changes integral means in a simple way:

$$I(K_k f, \delta/k) = \int \left|\frac{f'(r^k e^{ik\theta})}{f^{(k-1)/k}(r^k e^{ik\theta})}\right|^t r^{t(k-1)} d\theta.$$

As mentioned before, the capacity of the snowflake is bounded by the universal constant; hence $|f|$ can be bounded by a universal constant. Thus

$$I(K_k f, \delta/k) > \operatorname{const} I(f, \delta).$$

The function f_{n+1} is a composition of a (random) function ϕ_θ with Kf_n. The expectation of $I(f_{n+1}, 1/k^{n+1})$ conditioned on f_n is

$$\begin{aligned}\mathbb{E}\left[I(f_{n+1}, 1/k^{n+1}) \mid f_n\right] &= \int\int |\phi'_\theta(K_k f_n(re^{i\xi}))|^t |(K_k f_n)'(re^{i\xi})|^t d\xi d\theta \\ &= \int |(K_k f_n)'(re^{i\xi})|^t \int |\phi'(e^{-i\theta} K_k f_n(re^{i\xi}))|^t d\theta d\xi,\end{aligned}$$

where $r=\exp(1/k^{n+1})$. We know that $|K_k f_n(re^{i\xi})|-1<c/k$. By our choice of ϕ,

$$\int |\phi'(|K_k f_n(re^{i\xi})|e^{-i\theta})|^t d\theta > A\left(\frac{k}{c}\right)^{B(t)-\varepsilon}.$$

Thus,

$$\begin{aligned}\mathbb{E}\left[I(f_{n+1},1/k^{n+1})\right] &> A\left(\frac{k}{c}\right)^{B(t)-\varepsilon}\mathbb{E}\left[I(Kf_n,1/k^{n+1})\right]\\ &> A\left(\frac{k}{c}\right)^{B(t)-\varepsilon}\text{const}\,\mathbb{E}\left[I(f_n,1/k^n)\right].\end{aligned}$$

Applying this inequality n times we obtain

$$\mathbb{E}\left[I(f_n,1/k^n)\right] > \text{const}^n\left(\frac{k}{c}\right)^{n(B(t)-\varepsilon)}.$$

Then,

$$\frac{\log\mathbb{E}\left[I(f_n,1/k^n)\right]}{n\log k} > B(t)-2\varepsilon$$

for sufficiently large k. This completes the proof. □

5. Appendix: Example of an estimate

The main purpose of this section is to show that using conformal snowflakes it is not very difficult to find good estimates. Particularly it means that if one of the famous conjectures mentioned in the introduction is wrong, then it should be possible to find a counterexample.

In this section we will give an example of a simple snowflake and estimate its spectrum at $t = 1$. We could do essentially the same computation for other values of t, but $B(1)$ is of special interest because it is related to the coefficient problem and Littlewood conjecture (see [3] for details).

As a building block we use a very simple function: a straight slit map. We use the following scheme: first, we define a building block and this gives us the operator P. By (10) any positive function ν gives an estimate on the spectrum. To choose ν, we find the first eigenvector of the discretized operator P and approximate it by a rational function. We compute $P\nu$ using Euler's quadrature formula and estimate the error term. The minimum of $P\nu/\nu$ gives us the desired estimate of $\beta(1)$. For $t = 1$ we explain how one can rigorously estimate the error term in the computation of $P\nu/\nu$. For other values of t we give approximate values (computed with less precision) without any estimates of the error terms. Rigorous estimates can be found in a separate paper [2].

5.1. *Single slit domain.* We use straight slit functions. First, we define the basic slit function

$$\phi(z,l) = \phi_l(z) = \mu_2\left(\frac{\sqrt{\mu_1^2(zs)+l^2/(4k+4)}}{\sqrt{1+l^2/(4l+4)}}\right), \tag{11}$$

where s is a constant close to 1, μ_1 and μ_2 are the Möbius transformations that map $\mathbb{D}_-$ onto the right half-plane and its inverse:

$$\mu_1(z) = \frac{z-1}{z+1}, \qquad \mu_2(z) = \frac{z+1}{z-1}.$$

We also need the inverse function

$$\psi(z,l) = \psi_l(z) = \phi(z,l)^{-1}. \tag{12}$$

The function ϕ first maps $\mathbb{D}_-$ onto the right half-plane; then we cut off a straight horizontal slit starting at the origin and map it back. The image $\phi_l(\mathbb{D}_-)$ is $\mathbb{D}_-$ with a horizontal slit starting from 1. The length of the slit is l. The derivative of a slit map has a singularity at points that are mapped to 1. But if we take $s > 1$ then these singularities are not in $\mathbb{D}_-$. We set $s = 1.002$.

We study the snowflake generated by $\phi(z) = \phi_{73}(z)$ with $k = 13$ (numbers 13 and 73 are found experimentally). Figure 3 shows the image of the unit circle under f_3. Figure 4 shows the image of a small arc under $\bar{f}_3$ and three Green's lines.

First, we have to find the critical radius R such that $D_R \subset \psi^k(D_R)$. By symmetry of ϕ, the critical radius is the only positive solution of

$$\psi^k(x) = x.$$

This equation cannot be solved explicitly, but we can solve it numerically (we do not care about error term since we can take any greater value of R). The approximate value of R is 76.1568. To be on the safe side we fix $R = 76.2$. The disc takes just a small portion of $\psi(D_R)$ which means that there is a huge overkill in the inequality (7).

By (10) any positive function ν gives a lower bound of spectrum. And this estimate is sharp when ν is the main eigenfunction of P. So we have to find an "almost" eigenfunction of P.

5.2. *Almost eigenfunction of operator P.* Even for such a simple building block we cannot find the eigenfunction explicitly. Instead we look for some sort of approximation. The first idea is to substitute integral operator P by its discretized version. Here we use the most simple and quite crude approximation.

Choose sufficiently large N and M. Let $r_n = 1 + (R-1)n/N$ and $\theta_m = 2\pi m/M$. Instead of P we have an $N \times N$ matrix with elements

$$P_{n,n'} = \sum_m r_n^{1-t(k-1)/k} \frac{|\phi'(r_n^{1/k} e^{i\theta_m})|^t}{|\phi(r_n^{1/k} e^{i\theta_m})| M},$$

where the sum is over all m such that

$$\left| |\phi(r_n^{1/k} e^{i\theta_m})| - r_{n'} \right| < \frac{R-1}{2N}.$$

This defines the discretized operator P_N. Let λ_N and V_N be the main eigenvalue and the corresponding eigenvector. *A priori*, λ_N should converge to $k^{\beta(t)}$, but it is not easy to prove and not clear how to find the rate of convergence. But this crude estimate gives us the fast test whether the pair ϕ and k defines a snowflake with large spectrum or not (this is how we found $k = 13$ and $l = 73$).

Instead of proving convergence of λ_N and estimating the error term we will study V_N which is the discrete version of the eigenfunction. We approximate V_N by a rational function of relatively small degree (in our case 5), or by any other simple function. In our case we find the rational function by the linear least square fitting. In any way we get a nice and simple function ν which is supposed to be close to the eigenfunction of P.

We note that the procedure by which we obtained ν is highly non rigorous, but that does not matter since as soon as we have some explicit function ν we can plug it into P and get the rigorous estimate of β.

In our case we take $N = 1000$ and $M = 500$. The logarithm of the first eigenvalue is 0.2321 (it is 0.23492 if we take $s = 1$). Figure 6 shows a plot with coordinates of the first eigenvector.

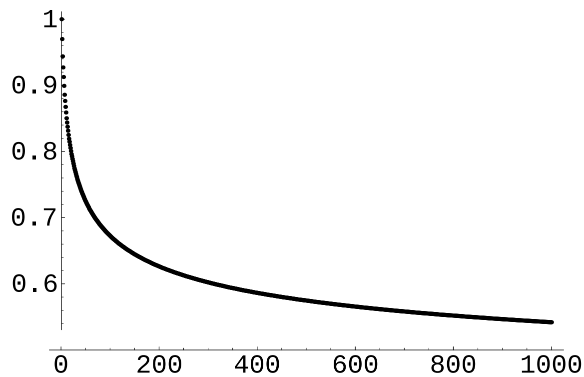

Figure 6. Coordinates of the eigenvector

We scale this data from $[1, 1000]$ to the interval $[1, R]$ and approximate by a rational function ν:

$$\nu(x) = (7.1479 + 8.9280x - 0.07765x^2 + 1.733 \times 10^{-3}x^3 \\ - 2.0598 \times 10^{-5}x^4 + 9.5353 \times 10^{-8}x^5)/(2.7154 + 13.2845x).$$

Figure 7 shows the plot of ν.

We compute $P\nu$ using the Euler quadrature formula based on the trapezoid quadrature formula

$$\int_0^{2\pi} f(x)dx \approx S_\varepsilon^n(f) = S_\varepsilon(f) - \sum_{k=1}^{n-1} \gamma_{2k}\varepsilon^{2k}\left(f^{(2k-1)}(2\pi) - f^{(2k-1)}(0)\right),$$

where $S_\varepsilon(f)$ is a trapezoid quadrature formula with step ε and $\gamma_k = B_k/k!$ where B_k is the Bernoulli number.

The error term in the Euler formula is

$$-\gamma_{2n} \max f^{(2n)}\varepsilon^{2n}2\pi. \tag{13}$$

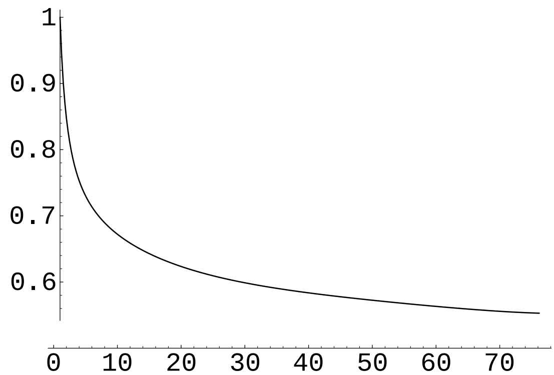

Figure 7. An “almost” eigenfunction

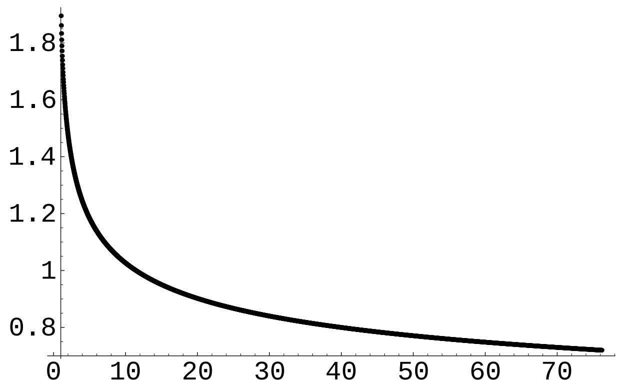

Figure 8. Values of $I(r)$

In our case function f is periodic and terms with higher derivatives vanish. This means that we can use (13) for any n as an estimate of the error in the trapezoid quadrature formula.

Let $\varepsilon = \pi/5000$. A relatively simple estimate of derivatives (see [2] for details) shows that the error term in this case is less than 0.0034.

Next we have to estimate the modulus of continuity with respect to r. Explicit computations can show that if we compute values $I(r_1)$ and $I(r_2)$ (with precision 0.0034) then the minimum of $P(\nu)/\nu$ on $[r_1, r_2]$ is at least

$$r_1^{1/k}(\min\{I(r_1), I(r_2)\} - 0.0034 - 0.0131(r_2 - r_1)r^{1/k-1})/\nu(r_1). \tag{14}$$

We take 3000 equidistributed points on $[1, R]$ and compute $I(r)$ at these points. The data for $I(r)$ is shown in Figure 8. Applying the error estimate (14) we find a rigorous estimate from below of $P\nu/\nu$. The minimum of $P\nu/\nu$ is at least 1.8079 which means that

$$\beta(1) > 0.2308.$$

Figure 9 shows the plot of $\log(P\nu/\nu)/\log k$.

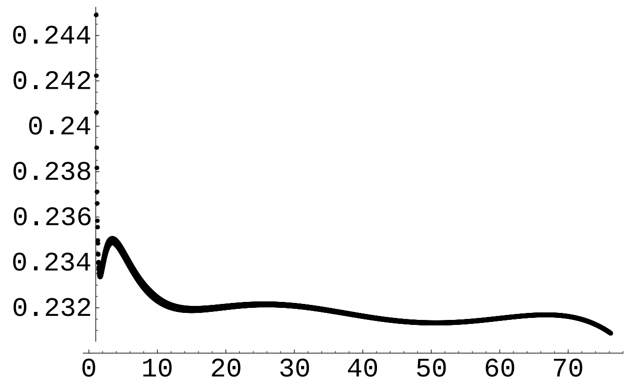

Figure 9. Plot of $\log(P\nu/\nu)/\log k$

5.3. *Estimates of spectrum for other values of t.* We also computed the lower bound on the spectrum of the same snowflake for other values of t. Below are given base 13 logarithms of eigenvalues of the discretized operator P ($N = 1000$, $M = 400$), lower bounds of $\log(P\nu/\nu)/\log k$, $t^2/4$ and the upper bound of the universal spectrum from [7], [10]. For values of t close to zero we cannot find a function that gives us a positive lower bound.

t	$\log_{13}\lambda$	$\beta(t) >$	$t^2/4$	$\beta(t) <$
-2.0	0.6350	0.56	1	1.218
-1.8	0.5348	0.48	0.81	1.042
-1.6	0.4395	0.39	0.64	0.871
-1.4	0.3502	0.31	0.49	0.706
-1.2	0.2678	0.220	0.36	0.549
-1.0	0.1936	0.152	0.25	0.403
-0.8	0.1290	0.0925	0.16	0.272
-0.6	0.0756	0.0430	0.09	0.159
-0.4	0.0353	0.0050	0.04	0.072
-0.2	0.0100	0	0.01	0.0179
0.2	0.0105	0	0.01	0.031
0.4	0.0387	0.0280	0.04	0.184
0.6	0.0858	0.0795	0.09	0.276
0.8	0.1515	0.1505	0.16	0.368
1.0	0.234	0.234	0.25	0.460
1.2	0.334	0.332	0.36	0.613
1.4	0.448	0.442	0.49	0.765
1.6	0.576	0.570	0.64	0.843
1.8	0.713	0.698	0.81	0.921
2.0	0.859	0.821	1	1

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E-mail address: dbeliaev@math.princeton.edu
DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, FINE HALL, WASHINGTON RD., PRINCETON, NJ 08544, UNITED STATES

E-mail address: Stanislav.Smirnov@unige.ch
UNIVERSITÉ DE GENÈVE, SECTION DE MATHÉMATIQUES, 2-4, RUE DU LIÈVRE, C.P. 64, CH-1211 GENÈVE 4, SWITZERLAND

http://www.unige.ch/∼smirnov/

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