Random Plane Waves

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Berry's conjecture

In 1977 M. Berry conjectured that high energy eigenfunctions in the chaotic case have statistically the same behaviour as random plane waves. (Figures from Bogomolny-Schmit paper)

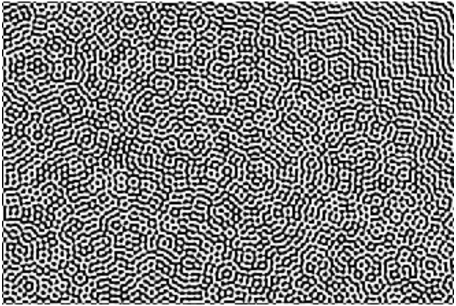


Figure : Nodal domains of an eigenfunction of a stadium

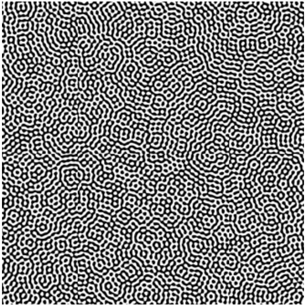


Figure : Nodal domains of a random plane wave

Random Plane Wave

There are several ways to define random plane waves with energy $E = k^2$:

- Naive definition

$$\Psi_n(z) = \Re\left(\sum_{j=1}^{n} e^{k(\theta_j, z)+\phi_j}\right)$$

  where θ_j are uniform random directions and ϕ_j are random phases. Random plane wave is the limit as n tends to infinity.

- Rigorous definition

$$\Psi(r, \theta) = \sum C_n J_{|n|}(kr) e^{in\theta}$$

  where $C_n = \overline{C}_{-n}$ are independent Gaussian random variables and J_n are Bessel functions.

Random Plane Wave

- One can think that random plane wave is the 2-d Fourier transform of the white noise on the unit circle. To make it rigorous we introduce $L_s^2(\mathbb{T})$ – the Hilbert space of L^2 functions on the unit circle that satisfy symmetry condition $\phi(-z) = \overline{\phi(z)}$. We define H to be Fourier transform of L_s^2 with scalar product inherited from L^2. This space consist of real analytic functions satisfying Helmholtz equation. Random plane wave is

$$\sum C_n \Phi_n$$

  where $\{\Phi_n\}$ is any orthonormal basis in H and C_n are independent Gaussians.

Naive definition corresponds to the approximation by δ-measures, the second to the ortjonormal basis of x^n in $L^2(\mathbb{T})$.

Random Plane Wave and Spherical Harmonic

Simple computation shows that the random plane wave can be described as the unique isotropic Gaussian field with covariance function $J_0(k|z - w|)$.

Spherical harmonics of degree n form a $2n + 1$ dimensional space of eigenfunctions of Laplacian on the sphere. Random spherical harmonic is the Gaussian vector in this space. It follows from work of Zelditch that the Gaussian plane wave is the scaling limit of the Gaussian spherical harmonic.

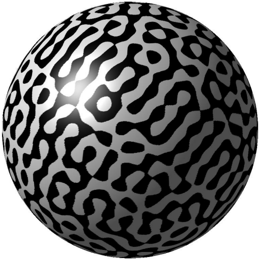


Figure : Spherical harmonic of degree 40. Picture by Alex Barnett

Band-Limited Functions

Let $(\mathcal{M}, g)$ be a compact Riemanian manifold, eigenfunctions of Laplacian form an orthonormal basis in L^2. Let t_i be the square roots of eigenvalues $0 \leq t_1 \leq t_2 \ldots$

$$\Delta\phi_i + t_i^2\phi_i = 0$$

Define band-limited function for $\alpha \in (0, 1)$

$$f_{\alpha,T} = \sum_{\alpha T < t_i < T} C_i\phi_i$$

For $\alpha = 1$

$$f_{\alpha,T} = \sum_{T - o(T) < t_i < T} C_i\phi_i$$

Deterministic Results

Some universal estimates are known for eigenfunctions of Laplacian.

Theorem

Nodal set for random plane wave forms a c/k-net where c is an absolute constant. Nodal set for spherical harmonic forms a c/n-net.

Theorem

Every nodal component contains a disc of radius c/k (or c/n) where c is an absolute constant.

Deterministic Results

The length of nodal set for spherical harmonic is relatively easy to compute using integral formulas due to Poincaré and Kac-Rice.

Theorem

There is a constant c such that for every spherical harmonic g of degree n such that

$$\frac{n}{c} < L(g) < cn$$

where $L(g)$ is the length of nodal set.

The lower bound is correct for every smooth Riemann surface (with n replaced by $\sqrt{\lambda}$). The upper bound is proven for real-analytic surfaces by Donelly and Fefferman.

Nodal Lines of Gaussian Spherical Harmonic

Theorem (Bérard, 1985)

For Gaussian spherical harmonic g_n of degree n

$$\mathbb{E}L(g_n) = \pi\sqrt{2\lambda_n} = \sqrt{2}\pi n + O(1)$$

With more careful analysis of Kac-Rice formula it is possible to compute variance

Theorem (I.Wigman, 2009)

For Gaussian spherical harmonic g_n of degree n

$$\mathrm{Var}\, L(g_n) = \frac{65}{32}\ln(n) + O(1)$$

Number of Nodal Domains

In the deterministic case Courant's theorem gives that the number of nodal domains $N(g_n) < n^2$. In 1956 Plejel improved the upper bound to $0.69n^2$. For $n > 2$ Lewy constructed spherical harmonic with two or three nodal domains, so there is no non-trivial deterministic lower bound.

The main problem: this is a non-local quantity.

Theorem (Nazarov and Sodin, 2007)

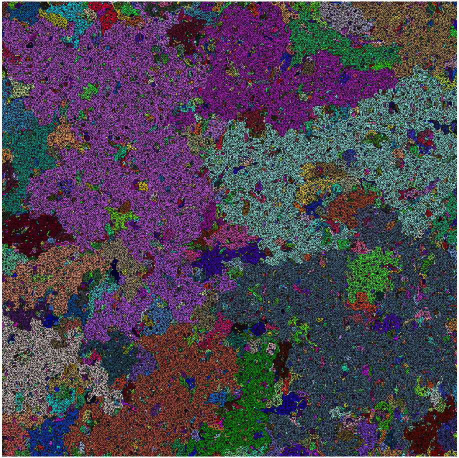
Let g_n be Gaussian spherical harmonic of degree n. Then there is a positive constant a such that

$$\mathbb{P}\left\{\left|\frac{N(g_n)}{n^2} - a\right| > \epsilon\right\} \leq C(\epsilon)e^{-c(\epsilon)n}$$

where $C(\epsilon)$ and $c(\epsilon)$ are positive constant depending on ϵ only.

Nodal Domains

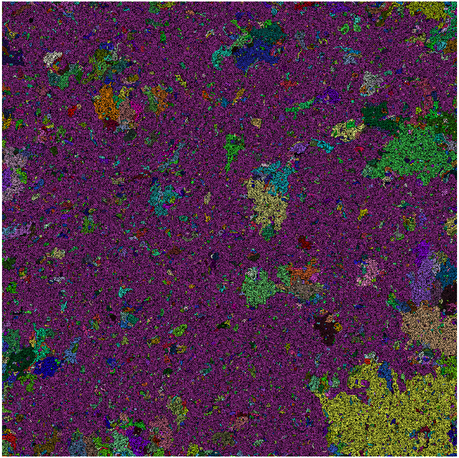
All positive nodal domains of a random plane wave.



Picture by T. Sharpe.

Nodal Domains

All negative nodal domains of a random plane wave.



Picture by T. Sharpe.

Nodal Domains Size Distribution

Theorem (B.–Wigman)

There is a limiting distribution of the nodal domain areas and nodal line lengths. This distribution function is strictly increasing starting from the lowest possible area.
The same is true for band-limited functions

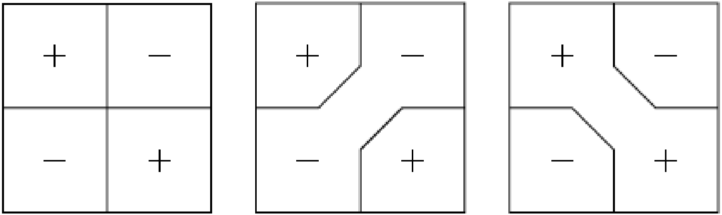
Theorem (Sarnak–Wigman)

There is a limiting distribution for the topology and nesting of the nodal domain.
The same is true for band-limited functions.

Proofs are combination of Kac-Rice, ergodicity and explicit constructions involving Lax-Malgrange approximation.

Bogomolny-Schmit Percolation Model

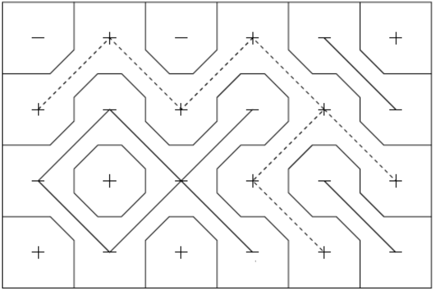
They proposed think that the nodal lines form a perturbed square lattice



Picture from Bogomolny-Schmit paper.

Bogomolny-Schmit Percolation Model

Using this analogy we can think of the nodal domains as percolation clusters on the square lattice. This leads to the conjecture that Nazarov-Sodin constant is $(3\sqrt{3}-5)/\pi \approx 0.0624$



Picture from Bogomolny-Schmit paper.

Bogomolny-Schmit Percolation Model

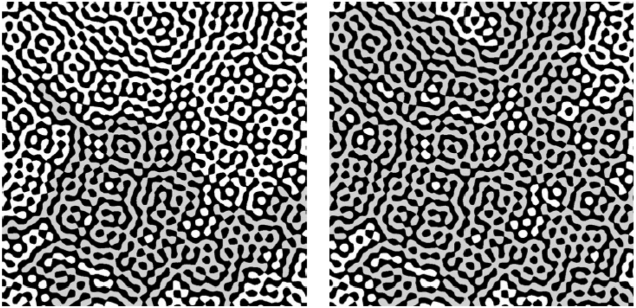


Figure 4. Left: nodal domains of a random wavefunction. Right: level domains of the same function with $\varepsilon = 0.03$. In the both figures the largest connected clusters are highlighted.

Picture from Bogomolny-Schmit paper.

Bogomolny-Schmit Conjecture

Bogomolny and Schmit conjectured that this critical bond percolation on the square lattice gives a good description of nodal domains. Based on this they predicted that

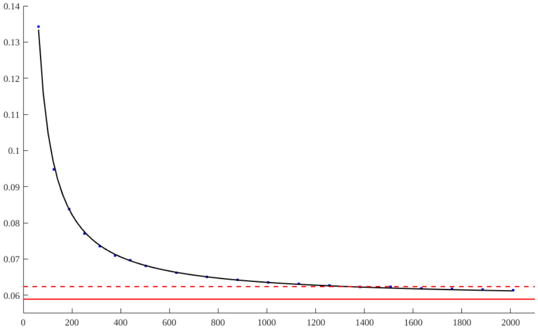
$$\frac{\mathbb{E}N(E)}{\bar{N}} = \frac{3\sqrt{3}-5}{\pi} \approx 0.0624$$

$$\frac{\mathrm{Var}N(E)}{\bar{N}} = \frac{18}{\pi^2} + \frac{4\sqrt{3}}{\pi} - \frac{25}{2\pi} \approx 0.0502$$

These conjectures are based on percolation cluster densities. Kleban claims that that one of the assumptions used in the derivation of the second formula is wrong and the correct answer should be approximately 2.085 times greater.

Numerical Results

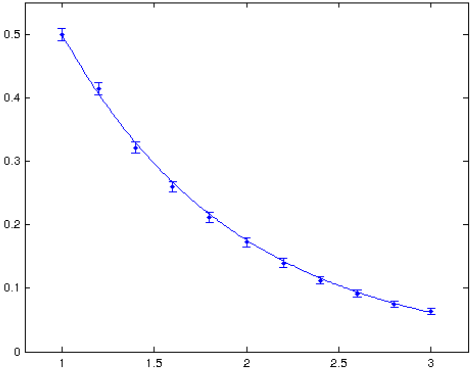
Several people (Nastasescu, Barnett, Konrad, Kereta, Sharpe) performed computer experiments with random plane waves. Figure by T. Sharpe.



The nodal domain density is 0.0589 which is 6% below Bogomolny-Schmit prediction.

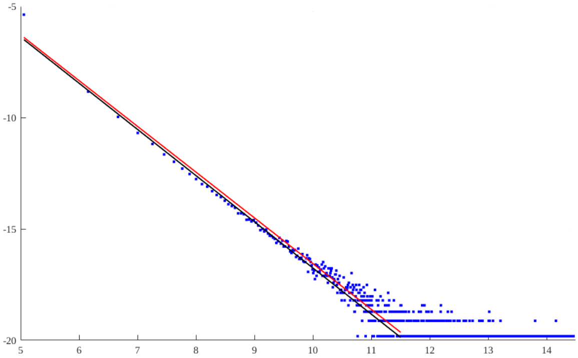
Universal Observable

Note that the nodal domain density is a universal quantity, it is the same for all surfaces. From percolation point of view this is a non-universal quantity. For universal (for percolation) observable match is much better. Crossing probability (by Z. Kereta)



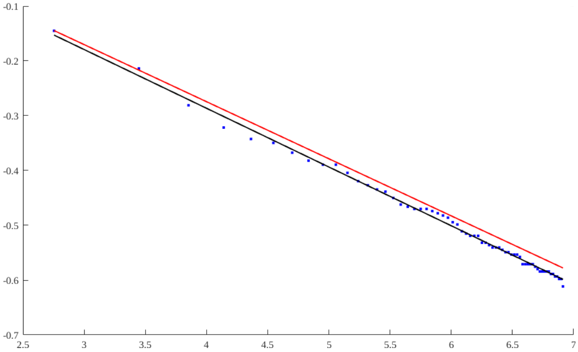
Universal Observable

Other universal also match very well The probability that the percolation cluster has area n is of order $n^{-\tau}$ where τ is Fisher constant $\tau = 187/91 \approx 2.055$. For nodal domains we have exponent 2.075 (T. Sharpe)



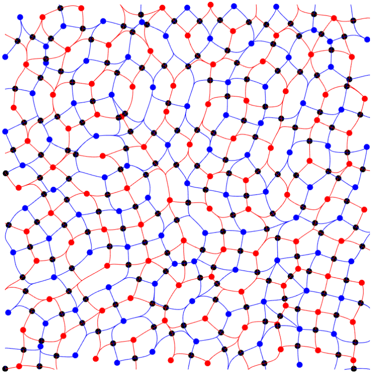
Universal Observable

Other universal also match very well The probability that the cluster containing origin has radius at least R is of order $R^{-\alpha}$ where α is one-arm exponent $\alpha = 5/48 \approx 0.104$. For nodal domains we have exponent 0.107 (T. Sharpe)



Alternative Percolation Model

We propose to consider bond percolation on a random graph generated by the random plane wave. The nodes of the graph are local maxima and the edges are gradient streamlines passing through saddles. Simulations by T. Sharpe



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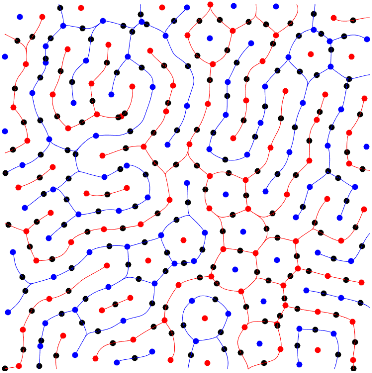

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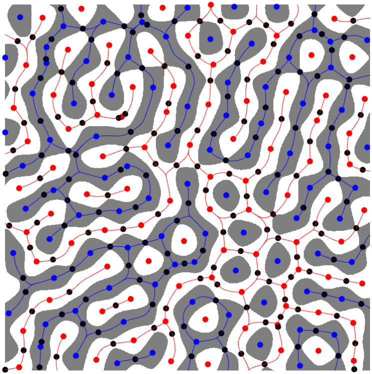
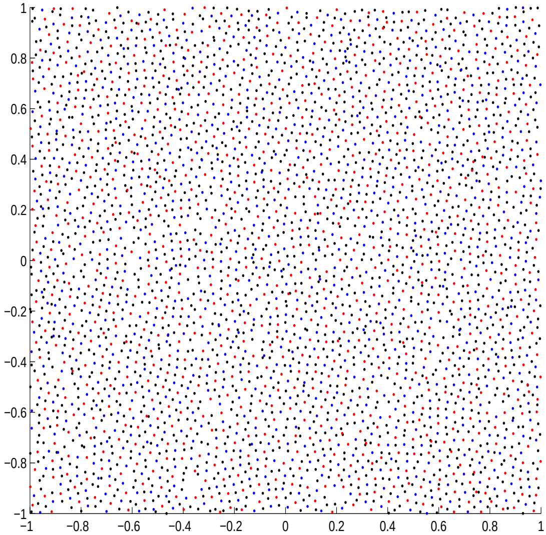


Critical points



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