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ON LITTLEWOOD'S CONSTANTS

D. BELIAEV AND S. SMIRNOV

Abstract

In two papers Littlewood studied seemingly unrelated constants: the best α such that for any polynomial f of degree n the areal integral of its spherical derivative is at most $const \cdot n^{\alpha}$, and the extremal growth rate β of the length of Green's equipotentials for simply connected domains. We show that these two constants coincide, thus greatly improving known estimates on α.

1. *Introduction*

In this paper we study the growth rate as $n \to \infty$ of the quantity

$$A_n = \sup \int_{\mathbb{D}} \frac{|g'|}{1+|g|^2} dm,$$

where supremum is taken over all polynomials g of degree n, $\mathbb{D}$ is the unit disc $\{|z| < 1\}$, and m denotes two-dimensional Lebesgue measure. We are interested in the best α such that $A_n \lesssim n^{\alpha}$ (which means that for every $\varepsilon > 0$ there is a constant C_ε with $A_n \le C_\varepsilon n^{\alpha+\varepsilon}$). In [**12**] Littlewood observed that $0 \le \alpha \le 1/2$ and conjectured that $\alpha < 1/2$. The problem of determining the best possible α appears under the number **4.18** in Hayman's problem list [**9**].

It is easy to show that there is a constant c such that for any rational function g of degree n

$$\int_{\mathbb{D}} \frac{|g'|}{1+|g|^2} dm \le c\sqrt{n}.$$

Note that integrand is a modulus of the spherical derivative g'_σ of g (i.e. derivative with respect to spherical metric) and that in $\mathbb{D}$ the spherical measure dm_σ is comparable to the Lebesgue measure dm. So our integral can be estimated by

$$\int_{\mathbb{D}} |g'_\sigma| dm_\sigma \le \int_{\mathbb{C}} |g'_\sigma| dm_\sigma \le \left(\int_{\mathbb{C}} |g'_\sigma|^2 dm_\sigma\right)^{1/2} \left(\int_{\mathbb{C}} dm_\sigma\right)^{1/2}$$
$$= (2\pi n)^{1/2}(2\pi)^{1/2} = 2\pi\sqrt{n},$$

here we use that rational function of degree n maps complex sphere to itself n-to-1 so the area of the image is n times bigger than the area of the sphere. In particular this argument shows that $\alpha \le 1/2$.

Littlewood's conjecture was proved in [**15**] by Lewis and Wu, who improving upon the work [**6**] of Eremenko and Sodin obtained an explicit upper estimate $\alpha < 1/2 - 2^{-264}$. Later Eremenko in [**5**] obtained a positive lower bound on α.

Following works by Eremenko and Sodin [**6**] and Lewis and Wu [**15**] we exploit connection between this problem and the extremal behavior of harmonic measure.

Our main result is that α is related to the growth rate of Green's lines length. In the case of simply connected domains Ω we define β_Ω as

$$\limsup_{\varepsilon \to 0} \frac{\log \operatorname{length}\{z : G(z) = \varepsilon\}}{\log 1/\varepsilon},$$

where G is the Green's function with pole at infinity and we define

$$\beta = \sup \beta_\Omega,$$

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where supremum is taken over all simply connected domains Ω. In the non simply connected case one needs more elaborate definition and we use the multifractal analysis technique.

For a given domain Ω we define the *packing spectrum* $\pi_\Omega(t)$ as

$$\sup\left\{q:\ \forall\delta>0\ \exists\ \delta-\text{packing }\{B\}\ \mathit{with}\ \sum \operatorname{diam}(B)^t\omega(B)^q\ \geq 1\right\},$$

where ω is the harmonic measure in Ω and δ-packing is a collection of disjoint open sets whose diameters do not exceed δ. Note that this definition is valid for any domain with compact boundary and for $t=1$ it is analogous to β_Ω (see [**16**] for the proof of $\beta_\Omega=\pi_\Omega(1)$ for simply connected Ω) .

We define the *universal spectrum* $\pi(t)$ as the supremum of $\pi_\Omega(t)$ over all planar domains Ω with compact boundary.

The main result of this paper is the following theorem:

Main Theorem. *For any positive ε there exists a constant $c=c(\varepsilon)$ such that*

$$A_n\leq cn^{\pi(1)+\varepsilon}.$$

Equivalently

$$\alpha\leq\pi(1).$$

Remark 1. Our proof actually implies that for $t\in[0,2]$

$$\int_{\mathbb{D}}\left(\frac{|g'|}{1+|g|^2}\right)^t\leq cn^{\pi(2-t)+\varepsilon}.$$

Of interest to us are also $\pi_p(t)$ and $\pi_{p,sc}(t)$ which are suprema of $\pi_\Omega(t)$ over all domains of attraction to infinity for polynomial mappings and simply connected domains of attraction to infinity for polynomial mappings correspondingly (see [**3**] for background on complex dynamics). It is clear that $\pi_{p,sc}\leq\pi_p\leq\pi$, but a priori they might differ.

In [**5**] Eremenko essentially proved that $\pi_{p,sc}(1)\leq\alpha$ (he works under assumption that polynomials are hyperbolic, but it can be easily avoided). Binder, Makarov, and Smirnov in [**1**] showed that $\pi_{p,sc}(t)=\pi_p(t)$ for $t>0$. Recently Binder and Jones announced the proof of the identity $\pi_p(t)=\pi(t)$, which together with our theorem completes the circle:

$$\alpha\leq\pi(1)=\pi_p(1)=\pi_{p,sc}(1)\leq\alpha.$$

There is yet another growth (or rather decay) rate that is related to α and was also studied by Littlewood. The growth rate γ of coefficients of univalent functions in $\mathbb{D}^-$ is defined by

$$\gamma:=\sup_\phi\limsup_{n\to\infty}\frac{\log|b_n|}{\log n}+1,$$

where the first supremum is taken over all functions $\phi(z)=z+\sum_{n=1}^\infty b_nz^{-n}$ that are univalent in $\mathbb{D}^-$. Littlewood proved [**13**, **14**] that $\beta\geq\gamma$. Much later Carleson and Jones [**4**] showed that $\gamma=\beta$. Summing it all up, we arrive at:

Corollary.

$$\alpha=\beta=\gamma=\pi(1)=\pi_{p,sc}(1).\tag{1.1}$$

The corollary uses aforementioned (yet unpublished) result of Binder and Jones. Note that a well-known conjecture (see [**4**, **16**, **11**, **17**]) states that $\pi(t)=(2-t)^2/4$ for $|t|\leq2$, in particular $\alpha=\beta=\gamma=1/4$. The best published estimates for β are

$$0.17\overset{[17]}{<}\beta\overset{[8]}{<}0.4884,$$

so the same estimates hold for α, which is a significant improvement over previously known

$$1.11\cdot10^{-5}\overset{[2]}{\leq}\alpha\overset{[14]}{\leq}1/2-2^{-264}.$$

Recently, Hedenmalm and Shimorin released preprint ([**10**]) with the estimate $\beta<0.46$. And authors recently obtained a computer assisted estimate from below: $\beta>0.23$ (in preparation).

1.1. *Connection to the value distribution of entire functions*

Before giving the proof we would like to remark that the reason for Littlewood's interest in this problem was the following striking corollary of his conjecture (see [**12**]):

LITTLEWOOD'S CONDITIONAL THEOREM. *Assume that $\alpha < 1/2$. If f is an entire function of order $0 < \rho < \infty$ then for any $0 < \theta < 1/2 - \alpha$ there is a "small" set S such that "almost all" roots of any equation $f(z) = w$ lie in S. Namely*

$$\frac{Area(S \cap B(R))}{Area(B(R))} = O\left(\frac{1}{R^{2\theta\rho}}\right), \qquad R \to \infty,$$

and for all w

$$\frac{\#\{z \in B(R) \setminus S : f(z) = w\}}{\#\{z \in B(R) : f(z) = w\}} = O\left(\frac{1}{R^{\rho(1/2-\alpha-\theta)}}\right), \qquad R \to \infty,$$

where $B(R)$ is a disc of radius R centered at the origin.

2. *Proof of The Main Theorem*

It is a standard fact (for details see [**16**] and [**7**]) that $\pi(t)$ is finite, convex, and strictly decreasing on $[0, 2]$, and hence for any small δ we can choose ε so small that

$$\pi(1 - 2\varepsilon) - \pi(1) < \delta.$$

We will also use a more elaborate fact (which follows from multifractal formalism and fractal approximation – see [**16**]), that there is a constant $\mathrm{const}(t, \varepsilon)$, such that for any disjoint collection of cubes $\{Q\}$ of size ≤ 1 one has

$$\sum_Q \omega(Q)^{\pi(t)} l(Q)^{t+\varepsilon} < \mathrm{const}(t, \varepsilon).$$

Let g be a polynomial of degree at most n and a_i its zeros. Consider a set where $|g|$ is big, i.e. $|g| \geq n$. We can easily estimate integral over this set by

$$\begin{aligned} \int\limits_{\mathbb{D}\cap\{|g|\geq n\}} \frac{|g'|}{1 + |g|^2} &\leq n^{-1} \int\limits_{\mathbb{D}} \frac{|g'|}{|g|} = n^{-1} \int\limits_{\mathbb{D}} |(\log g)'| \\ &= n^{-1} \int\limits_{\mathbb{D}} \left| \sum_{i=1}^{n} \frac{1}{z - a_i} \right| \leq n^{-1} \cdot 2\pi n = 2\pi. \end{aligned} \tag{2.1}$$

Now consider a complementary set where $|g|$ is small, which is contained inside the disc of radius $3/2$

$$\Omega := \{z : |g(z)| < n, |z| < \frac{3}{2}\} ,$$

and let $W = \{Q_j\}$ be a Whitney decomposition of Ω. We note that

$$d\mu(z) = 4n^{-1} \frac{|g'(z)|^2}{(1 + |g(z)|^2)^2} dxdy,$$

is the Riesz measure associated with the nonnegative subharmonic function

$$u = \frac{\log(1 + |g|^2)}{n} .$$

Then by Riesz representation theorem

$$\mu(B(z, r)) \leq \frac{c}{2\pi} \int\limits_0^{2\pi} (u(z + 2re^{i\theta}) - u(z))d\theta. \tag{2.2}$$

Hence for every cube Q_j we have

$$\mu(Q_j) \leq c\frac{\log(1 + n)}{n}.$$

Fix a cube Q_j such that $Q_j \cap \mathbb{D} \neq \emptyset$ and denote by ξ_j a point at $\partial\Omega$ such that $d(\xi_j, Q_j) \leq 2l(Q_j)$.

Then from (2.2)

$$\begin{aligned}\mu(Q_j) &\leq \frac{c}{2\pi}\int_0^{2\pi}(u(\xi_j+8l(Q_j)e^{i\theta})-u(\xi_j))d\theta \\ &\leq \max_{z\in B(\xi_j,8l(Q_j))} c[u(z)-u(\xi_j)]^+ .\end{aligned} \tag{2.3}$$

Denote by $G(z)$ the Green's function for $\mathbb{C}\setminus\overline{\Omega}$ with pole at infinity. Extend G to a continuous subharmonic function in $\mathbb{C}$ by setting $G=0$ on Ω.

By the maximum principle for domain $\mathbb{C}\setminus\overline{\Omega}$ we obtain $G(z)\geq\log|2z/3|$ for any $z\in\mathbb{C}\setminus\Omega$, hence

$$G(z)\geq\log\frac{4}{3}\qquad\text{for } |z|=2.$$

By the maximum principle for domain $2\mathbb{D}\setminus\overline{\Omega}$, we have

$$u(z)-u(\xi_j)\leq M_2G(z)\left(\log\frac{4}{3}\right)^{-1}\qquad\text{for } |z|\leq 2,$$

where $M_2=\max_{|z|=2}u(z)$.

If we let z_j be a center of Q_j and ω be a harmonic measure on $\mathbb{C}\setminus\overline{\Omega}$ with pole at infinity then by previous inequality and (2.3) we have

$$\begin{aligned}\mu(Q_j) &\leq \max_{z\in B(\xi_j,8l(Q_j))} c[u(z)-u(\xi_j)]^+ \\ &\leq c\left(\log\frac{4}{3}\right)^{-1} M_2 \max_{z\in B(\xi_j,8l(Q_j))} G(z)\leq cM_2\max_{z\in B(z_j,16l(Q_j))} G(z).\end{aligned}$$

By Harnack's inequality right hand side is less than

$$cM_2 3\int_{\partial B(z_j,32l(Q_j))} G(z)\frac{|dz|}{2\pi 32l(Q_j)},$$

which by Riesz representation formula equals to

$$cM_2\int_0^{32l(Q_j)}\frac{\omega(B(z_j,t))}{t}dt\leq cM_2\omega(B(z_j,32l(Q_j))).$$

So finally we have

$$\mu(Q_j)\leq \text{const } M_2\ \omega(B(z_j,32l(Q_j))). \tag{2.4}$$

By Schwarz's inequality we have

$$\begin{aligned}&\int_\Omega\frac{|g'|}{1+|g|^2}dxdy=n^{1/2}\sum_{Q_j\in W}n^{-1/2}\int_{Q_j}\frac{|g'|}{1+|g|^2}dxdy \\ &\leq\frac{1}{2}n^{1/2}\sum_{Q_j\in W}\left(\int_{Q_j}\frac{4|g'|^2}{n(1+|g|^2)^2}dxdy\right)^{1/2}\left(\int_{Q_j}dxdy\right)^{1/2} \\ &\leq\frac{1}{2}n^{1/2}\sum_{Q_j\in W}\mu(Q_j)^{1/2}l(Q_j)=\frac{1}{2}n^{1/2}\sum_{Q_j\in W}\mu(Q_j)^{1/2-\pi(1-2\varepsilon)}\mu(Q_j)^{\pi(1-2\varepsilon)}l(Q_j) \\ &\leq Cn^{1/2}\left(\frac{\log(1+n)}{n}\right)^{1/2-\pi(1-2\varepsilon)}\sum_{Q_j\in W}\mu(Q_j)^{\pi(1-2\varepsilon)}l(Q_j) \\ &\leq Cn^{\pi(1)+\delta}\sum_{Q_j\in D}\mu(Q_j)^{\pi(1-2\varepsilon)}l(Q_j),\end{aligned}$$

where D is the family of all dyadic squares with the side length less then 32 that intersect Ω, ε is a small positive number, and C is a constant. By (2.4) we can estimate the last sum

$$\begin{aligned} &Cn^{\pi(1)+\delta} \sum_{Q_j \in D} \mu(Q_j)^{\pi(1-2\varepsilon)} l(Q_j) \\ &\le C\, M_2 n^{\pi(1)+\delta} \sum_{k=1}^{\infty} \sum_{l(Q_j)=1/2^k} \omega(Q_j)^{\pi(1-2\varepsilon)} l(Q_j)^{(1-2\varepsilon)+\varepsilon} l(Q_j)^{\varepsilon} \\ &= C\, M_2 n^{\pi(1)+\delta} \sum_{k=1}^{\infty} \frac{1}{2^{k\varepsilon}} \sum_{l(Q_j)=1/2^k} \omega(Q_j)^{\pi(1-2\varepsilon)} l(Q_j)^{(1-2\varepsilon)+\varepsilon} \\ &\le C\, M_2 n^{\pi(1)+\delta} \sum_{k=1}^{\infty} \frac{1}{2^{k\varepsilon}} \operatorname{const}(\varepsilon) \le n^{\pi(1)+\delta} \operatorname{const}(\varepsilon) M_2. \end{aligned} \tag{2.5}$$

Now assume the following dichotomy: *For any* $\varepsilon > 0$ *there exists a constant* const(ε) *such that for any polynomial* g *of degree* n

$$M_2 \le \operatorname{const}(\varepsilon) n^{\varepsilon}, \tag{2.6}$$

or

$$|\Omega| \le 1/n. \tag{2.7}$$

Then if (2.6) holds then the desired estimate follows from (2.5):

$$\int_{\Omega} \frac{|g'|}{1+|g|^2} \le \operatorname{const} n^{\pi(1)+\varepsilon+\delta}.$$

But both ε and δ can be made arbitrary small we have the desired estimate.

If (2.7) holds then even better estimate follows from Schwartz' inequality:

$$\begin{aligned} \int_{\Omega} \frac{|g'|}{1+|g|^2} &\le \left(\int_{\Omega} \frac{|g'|^2}{(1+|g|^2)^2} \right)^{1/2} \left(\int_{\Omega} 1 \right)^{1/2} \\ &\le \left(\int_{\mathbb{C}} |g'_{\sigma}| \right)^{1/2} \sqrt{|\Omega|} \le \sqrt{2\pi n |\Omega|} \le \sqrt{2\pi}. \end{aligned}$$

Therefore it remains to prove the dichotomy.

Proof of the dichotomy. Assume that $M_2 > n^{\varepsilon}$. Recalling the definition $M_2 = \sup_{|z|=2} \log(1+|g|^2)/n$, we deduce that $\sup_{|z|=2}|g| > \exp(n^{1+\varepsilon})$ so the set Ω where $|g| < n$ cannot have big measure.

We can write g as $g = PQ$, where

$$P(z) = \lambda \prod_{|a_i|>4} (z-a_i), \quad Q(z) = \lambda \prod_{|a_i|\le 4} (z-a_i).$$

Let m be the degree of P, then

$$\log\left(|\lambda| \prod_{|a_i|>4} |a_i| \right) - m\log 2 \le \log|P(z)| \le \log\left(|\lambda| \prod_{|a_i|>4} |a_i| \right) + m\log 2,$$

when $|z| \le 2$. Since $|Q(z)| < 6^n$ for $|z| \le 2$, it follows that

$$\inf_{z\in\Omega} \log|P(z)| \ge \sup_{|z|=2} \log|P(z)| 3^{-n} \ge \log\left(\exp(n^{1+\varepsilon}) 3^{-n} 6^{-n}\right) \ge \frac{1}{2} n^{1+\varepsilon},$$

if n is sufficiently large. Since $\log|PQ| = \log|g| < n$ in Ω we can write

$$\log|Q(z)| \le \log|PQ| - \log|P| \le n - \frac{1}{2} n^{1+\varepsilon} \le -\frac{1}{4} n^{1+\varepsilon}, \; z \in \Omega$$

if n is large enough. Therefore, Ω is contained in the union of disks $\{z : |z-a_i| \le \exp(-n^{\varepsilon}/4)\}$. Hence

$$|\Omega| \le n\pi \exp(-\frac{1}{2} n^{\varepsilon}) \le 1/n \qquad \text{when } n \text{ is sufficiently large.}$$

This proves the dichotomy for $n > N(\varepsilon)$, for degree n bounded from above by $N(\varepsilon)$ the dichotomy is easy by compactness argument (and anyway, it suffices to prove the estimate for polynomials of sufficiently large degree). This completes the proof of the main theorem. $\square$

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Dmitri Beliaev
Kungliga Tekniska Högskolan
Inst. för Matematik
100 44 Stockholm, Sweden

beliaev@math.kth.se

Stanislav Smirnov
Kungliga Tekniska Högskolan
Inst. för Matematik
100 44 Stockholm, Sweden

stas@math.kth.se

Université de Genève
Section de Mathématiques
2-4, rue du Lièvre
Case postale 240
1211 Genève 24, Switzerland

Stanislav.Smirnov@math.unige.ch