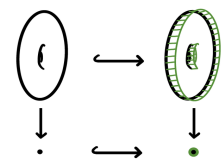
∞-Categories and Deformation Theory

Lecture 1

Fix a smooth and proper variety Z over the complex numbers. A (formal) deformation of Z to a local Artinian $\mathbb{C}$-algebra A is a pullback square

$$\begin{array}{ccc} Z & \longrightarrow & \widetilde{Z} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle flat} \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$



Two such deformations $\widetilde{Z}, \widetilde{Z}'$ are isomorphic if we can find an isomorphism $\widetilde{Z} \xrightarrow{\cong} \widetilde{Z}'$ over $\mathrm{Spec}(A)$ that restricts to the identity on Z. One important goal of deformation theory is:

Goal. Classify the deformations of Z over A.

Before discussing this difficult problem, we will first explore a much simpler one.

First order deformations. Consider the ring of dual numbers

$$A = \mathbb{C}[\epsilon]/\epsilon^2.$$

Deformations of Z to $\mathbb{C}[\epsilon]/\epsilon^2$ are called *first order deformations*. They are classified by the following theorem, which is an instance of Kodaira–Spencer theory [KS58]:

Theorem 1.1. *There is a canonical bijection*

$$\left\{\text{First order deformations of } Z\right\} \quad \cong \quad H^1(Z, T_Z)$$

To prove this result, we will need several basic facts from algebraic geometry. We will leave them as an exercise, which will serve as a reminder of several basic notions we will use later.

Exercise 1.2. *Let $\widetilde{Z}$ be a first order deformation of Z.*

(1) Show that $Z \to \widetilde{Z}$ induces a homeomorphism on underlying topological spaces;
(2) Show that if Z is affine, then so is $\widetilde{Z}$.

Exercise 1.3 (Infinitesimal lifting property)**.** *Let $A \to B$ be a smooth ring map, and assume $B = \widetilde{B}/I$ for some square-zero ideal $I \subset \widetilde{B}$. Show that there exists a lift*

$$\begin{array}{ccc} & & \widetilde{B} \\ & \nearrow & \downarrow \\ A & \longrightarrow & B. \end{array}$$

Exercise 1.4 (Detecting isomorphisms on special fibres)**.** *Let X_1, X_2 be two schemes which are flat and of finite type over $\mathbb{C}[\epsilon]/\epsilon^2$. A morphism $f : X_1 \to X_2$ over $\mathbb{C}[\epsilon]/\epsilon^2$ is an isomorphism if and only if the induced map $X_1 \times_{\mathbb{C}[\epsilon]/\epsilon^2} \mathbb{C} \to X_2 \times_{\mathbb{C}[\epsilon]/\epsilon^2} \mathbb{C}$ is an isomorphism.*

With these results, we can now prove the above classification of first order deformations.

Proof of Theorem 1.1. Let us pick a open cover $\mathfrak{U} = \{U_i\}$ of

$$Z = \bigcup_i U_i$$

by affine open subsets $U_i \cong \mathrm{Spec}(B_i) \subset Z$ with affine open intersections $U_{ij} \cong \mathrm{Spec}(B_{ij})$.

Our goal is to produce a cohomology class $x_{\tilde{Z}} \in H^1(Z, T_Z)$ for each first order deformation $\tilde{Z}$. To this end, fix a first oder deformation $\tilde{Z}$ as above. For each affine open $U_i \subset Z$, we use Exercise 1.2(1) to obtain an induced deformation

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{U_i} \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2). \end{tikzcd}$$

By Exercise 1.2(2), each restricted deformation $\widetilde{U_i} \cong \mathrm{Spec}(\widetilde{R_i})$ is also affine.
Tensoring the short exact sequence $0 \to (\epsilon) \to \mathbb{C}[\epsilon]/\epsilon^2 \to \mathbb{C} \to 0$ over $\mathbb{C}[\epsilon]/\epsilon^2$ with $\widetilde{R_i}$, we see that the flatness of $\widetilde{R_i}$ over $\mathbb{C}[\epsilon]/\epsilon^2$ implies that the kernel of $\widetilde{R_i} \to R_i$ is a square-zero ideal.

By Exercise 1.3, we can therefore pick a lift

$$\begin{tikzcd} & \widetilde{U_i} \arrow[dl, dashed] \\ U_i \arrow[r, "\mathrm{id}"] & U_i. \arrow[u] \end{tikzcd}$$

and obtain a map

$$\phi_i : \widetilde{U_i} \to U_i \times \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$$

Exercise 1.4 shows that ϕ_i is an isomorphism – we have trivialised the deformation on each affine U_i.

On each overlap $U_i \cap U_j \cong \mathrm{Spec}(R_{ij})$, the transition map $\theta_{ij} = \phi_i \phi_j^{-1}$ defines an automorphism

$$\theta_{ij} : R_{ij}[\epsilon]/\epsilon^2 \to R_{ij}[\epsilon]/\epsilon^2,$$

which is $\mathbb{C}[\epsilon]/\epsilon^2$-linear and compatible with the projection $R_{ij}[\epsilon]/\epsilon^2 \to R_{ij}$.

Such maps are simply given by $\mathbb{C}$-linear derivations $R_{ij} \to R_{ij}$, i.e. R_{ij}-linear maps $\Omega^1_{R_{ij}/\mathbb{C}} \to R_{ij}$, i.e. sections

$$\theta_{ij} \in \Gamma(U_{ij}, T_Z),$$

where T_Z is the tangent sheaf of Z.

Using that composition of automorphisms correspond to addition of the corresponding derivations, we check that $\theta_{ij} + \theta_{jk} + \theta_{ki} = 0$ for all i, j, k. We also check that if ϕ_i' are different trivialisations, then there is a secion $\alpha_i \in H^0(U_i, T_X)$ such that $\theta'_{ij} = \theta_{ij} + \alpha_i - \alpha_j$.

Hence we get a well-defined element $x_{\tilde{Z}}$ in the Čech cohomology group $\check{H}^1(\mathfrak{U}, T_Z) \cong H^1(Z, T_Z)$.

Exercise 1.5.

(1) Check that the element $x_{\tilde{Z}} \in H^1(Z, T_Z)$ does not depend on the chosen open covering;
(2) Produce a map from $H^1(Z, T_Z)$ to first order deformations of Z, and verify that it is an inverse to the above construction.

□

Towards higher order deformations. To classify higher order deformations of Z, we will refine $\mathrm{H}^*(Z, T_Z)$ to a *differential graded Lie algebra* $\mathfrak{g}_Z$, i.e. a chain complex with a bilinear bracket satisfying the Jacobi identity, antisymmetry, and the Leibniz rule, all in a graded sense.

Concretely, $\mathfrak{g}_Z$ is given by the *Dolbeault complex*

$$\mathfrak{g}_Z = \left(\mathcal{A}^{0,0}(T_Z) \to \mathcal{A}^{0,1}(T_Z) \to \mathcal{A}^{0,2}(T_Z) \to \ldots\right),$$

where $\mathcal{A}^{0,k}(T_Z)$ locally looks like

$$f \; d\overline{z}_{i_1} \wedge \ldots \wedge d\overline{z}_{i_k} \otimes \alpha.$$

The Lie bracket is obtained by wedging differential forms and taking the commutator of vector fields.

Miraculously, this d.g. Lie algebra in fact controls *all* (formal) deformations of Z. Indeed, we will see that for A a local Artin $\mathbb{C}$-algebra with maximal ideal $\mathfrak{m}_A$, there is a bijection

$$\left\{ \begin{array}{c} \text{Deformations } \widetilde{Z} \text{ of } Z \\ \text{over } \mathrm{Spec}(A) \end{array} \right\} \Big/ {\scriptstyle \begin{array}{c}\text{isomorphism}\\ \text{restricting to } \mathrm{id}_Z\end{array}} \cong \left\{ \begin{array}{c} \text{Maurer–Cartan elements} \\ x \in (\mathfrak{g}_Z)_{-1} \otimes \mathfrak{m}_A \,:\, dx + \frac{1}{2}[x,x] = 0 \end{array} \right\} \Big/ {\scriptstyle \begin{array}{c}\text{gauge}\\ \text{equivalence}\end{array}}.$$

Here $x, y \in (\mathfrak{g}_Z)_{-1} \otimes \mathfrak{m}_A$ are called *gauge equivalent* if there is some $a \in (\mathfrak{g}_Z)_0 \otimes \mathfrak{m}_A$ satisfying

$$y = x + \sum_{n=0}^{\infty} \frac{[a,-]^{\circ n}}{(n+1)!}([a,x] - da)$$

Other deformation problems. There are numerous other algebro-geometric objects Y over $\mathbb{C}$ whose infinitesimal deformations are governed by some d.g. Lie algebra $\mathfrak{g}_Y$, including subschemes, vector bundles, and representations. It is therefore natural to ask:

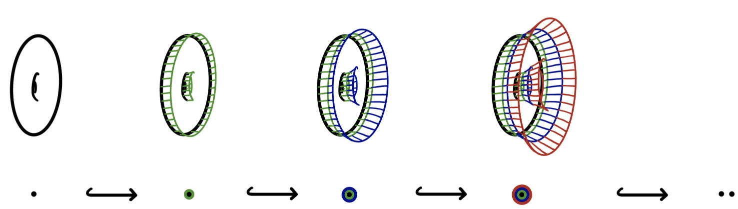
Question. Given an algebro-geometric object Y over $\mathbb{C}$, how can we construct the d.g. Lie algebra $\mathfrak{g}_Y$ controlling its infinitesimal deformations?

Unfortunately, many non-equivalent d.g. Lie algebras can control deformations of the same object, and it is not possible to functorially pick a preferred Lie algebra for a given deformation functor. On a more concrete note, this means obstruction classes are not functorial.

In a visionary letter [Dri] from 1988, Drinfel'd suggested that this issue would disappear once we also took *derived* infinitesimal deformations into account, i.e. deformations over simplicial local Artin $\mathbb{C}$-algebras

$$A \;=\; (\ldots \substack{\rightarrow\\\rightarrow\\\rightarrow} A_1 \rightrightarrows A_0)\,.$$

In these lectures, we will construct an equivalence between derived deformation functors and differential graded Lie algebras, and to use this equivalence to prove unobstructedness of Calabi–Yau varieties.



We will take a scenic route via Koszul–Moore duality, and learn some useful techniques in higher category theory along the way.

Prelude: Morita Theory

1.1. **Categorical Morita Theory.** Before discussing Koszul–Moore duality, an inherently higher categorical phenomenon, we will review Morita theory [Mor58], a good toy example which can be treated using only ordinary categories. It is centered around the following simple question:

Question. Given two associative rings R and S, is there an equivalence between the categories of left modules Mod_R and Mod_S?

If such an equivalence exists, then the rings R and S are said to be *Morita equivalent.* Isomorphic rings are clearly Morita equivalent, but the converse need not be true:

Proposition 1.6 (Morita functors)**.** *Let $Q \in \mathrm{Mod}_R$ be a left module over a ring R such that*

(1) Q is finite projective, i.e. a direct summand of $R^{\oplus n}$ for some n;
(2) Q is a generator, which means that the functor $\mathrm{Hom}_R(Q, -)$ is faithful.

Then R and $S = \mathrm{End}_R(Q)^{op}$ are Morita equivalent, which is witnessed by inverse equivalences

$$\widetilde{G} : \mathrm{Mod}_R \to \mathrm{Mod}_S, \quad M \mapsto \mathrm{Hom}_R(Q, M)$$
$$\widetilde{F} : \mathrm{Mod}_S \to \mathrm{Mod}_R, \quad N \mapsto Q \otimes_S N.$$

Before proving this claim, we give a simple exercise:

Exercise 1.7 (Examples of Morita equivalences)**.**

a) *Prove directly that for any ring R and any $n > 0$, the ring R is Morita equivalent to $M_n(R)$.*
b) *Find a ring R and a finite projective generator $Q \in \mathrm{Mod}_R$ such that $S = \mathrm{End}_R(Q)^{op}$ is not a matrix algebra.*

We present a categorical proof of Proposition 1.6 which we have learned from [Lur, Section 4.8]. While needlessly abstract, it will generalise well to ∞-categories of chain complexes and serve as good excuse to revise some basic categorical notions. We will implement the following strategy:

Strategy 1.8.

(0) Consider the functor $G : \mathrm{Mod}_R \to \mathrm{Mod}_{\mathbb{Z}}$ given by $M \mapsto \mathrm{Hom}_R(Q, M)$;
(1) Construct the associative ring $S = \mathrm{End}_R(Q)^{op}$ from the functor G;
(2) Lift G to a functor $\widetilde{G} : \mathrm{Mod}_R \to \mathrm{Mod}_S$ by exhibiting an S-module structure on each $\mathrm{Hom}_R(Q, M)$;
(3) Show that $\widetilde{G}$ is an equivalence.

1.2. **Properties of Functors.** We begin by reformulating the algebraic conditions imposed in Proposition 1.6 on $Q \in \mathrm{Mod}_R$ in terms of the associated functor $G : \mathrm{Mod}_R \to \mathrm{Mod}_{\mathbb{Z}}$. We treat the "finite" and the "projective" part in (1) separately, and start with the former.

Compactness. The categorical notion of compactness aims to capture the smallness of a given object X by asserting that it cannot be "spread out" arbitrarily.

For example, given a diagram $Y_0 \to Y_1 \to \ldots$, any map from a small object X to the sequential colimit $\mathrm{colim}_i\, Y_i$ (which we might think of as an increasing union) should factor through some Y_i.

In fact, we will also want to take slightly more general diagrams into account:

Definition 1.9 (Filtered categories)**.** A category I is *filtered* if it is nonempty and

a) any two objects x, y map into a third object z via morphisms $x \to z$, $y \to z$;
b) for all parallel morphisms $f, g : x \rightrightarrows y$ in $\mathcal{C}$, there exists $h : y \to z$ with $h \circ f = h \circ g$.

A *filtered colimit* in a category $\mathcal{C}$ is a colimit over a diagram $D : I \to \mathcal{C}$, where I is filtered.

Exercise 1.10. *Establish the following facts:*

(1) The category $\mathbb{N} = (\bullet \to \bullet \to \ldots)$ is filtered; hence sequential colimits are filtered;
(2) The product of filtered categories is filtered;
(3) The category $\bullet\ \ \bullet$ is not *filtered, and neither is Δ^{op}, the opposite of the category of nonempty finite linearly ordered sets.*

We can explicitly compute filtered colimits in the category of sets:

Exercise 1.11 (Filtered colimits of sets)**.** *Given a diagram $D : I \to \mathrm{Set}$ with I a small filtered category, show that $\mathrm{colim}_{i\in I} D(i)$ is given by the set $\coprod_{i\in I} D(i)/_{\cong}$, where $\cong$ is the equivalence relation identifying $a \in D(i), b \in D(j)$ if there are arrows $f : i \to k$, $g : j \to k$ with $D(f)(a) = D(g)(b)$.*

Exercise 1.12 (Limits of sets)**.** *Given a diagram $D : I \to \mathrm{Set}$ with I small, write down its limit.*

We will often need the following important fact:

Exercise 1.13 (Filtered colimits and finite limits in Set)**.**

a) Given a diagram $D : I \times J \to \mathrm{Set}$ with I a small filtered category and J a category with finitely many objects and morphisms, the following canonical arrow is an isomorphism:

$$\operatorname*{colim}_{i\in I}\left(\lim_{j\in J} D(i,j)\right) \xrightarrow{\cong} \lim_{j\in J}\left(\operatorname*{colim}_{i\in I} D(i,j)\right).$$

b) Show that filtered colimits generally do not commute with limits in Set*.*
c) Show that in Set^{op}*, filtered colimits need not commute with finite limits.*

Filtered colimits and finite limits also commute in categories that are sufficiently similar to sets. To make this precise, we need several notions.

Notation 1.14. Given a category I, the *right cone* $I^{\triangleright}$ is obtained from I by adding a new object 1 and a unique morphism from every $i \in I$ to the new object 1.

Definition 1.15. Let I be a category. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ preserves and reflects colimits of shape I if $D^{\triangleright} : I^{\triangleright} \to \mathcal{C}$ is a colimit diagram if and only if this is true for $U \circ D^{\triangleright} : I^{\triangleright} \to \mathcal{D}$. A similar definition applies to limits.

Using that faithful functors reflect isomorphisms (which we establish in Proposition 1.31 below), we can deduce the following basic fact from Exercise 1.13:

Corollary 1.16. *Let $U : \mathcal{C} \to \mathrm{Set}$ be a faithful functor which preserves and reflects finite limits and filtered colimits. Then finite limits commute with filtered colimits in $\mathcal{C}$.*

Exercise 1.17. *Show that for any ring R, the forgetful functor $U : \mathrm{Mod}_R \to \mathrm{Set}$ satisfies the assumptions of Corollary 1.16. Hint: equip the colimit of sets $\mathrm{colim}_{i\in I}(U \circ D)(i)$ constructed in Exercise 1.12 with the structure of an R-module.*

We can now give a categorical notion of smallness:

Definition 1.18. An object X in a locally small category $\mathcal{C}$ is called *compact* if the functor $\mathrm{Map}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathrm{Set}$ preserves filtered colimits.

Using Exercise 1.13, we can prove an intuitive closure property for compact objects:

Corollary 1.19. *Finite colimits of compact objects in a category $\mathcal{C}$ are compact.*

Proof. For any finite diagram $D : J \to \mathcal{C}$ which admits a colimit in $\mathcal{C}$, we have a natural isomorphism of functors $\mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_{j\in J} D(j), -) \xrightarrow{\cong} \lim_{j\in J} \mathrm{Map}_{\mathcal{C}}(D(j), -)$. For any filtered diagram $D' : I \to \mathcal{C}$, compactness of all $D(j)$ and Exercise 1.13 implies:

$$\begin{aligned} \mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_{j\in J} D(j), \mathrm{colim}_{i\in I} D'(i)) \quad &\cong \lim_{j\in J} \mathrm{colim}_{i\in I} \mathrm{Map}_{\mathcal{C}}(D(j), D'(i)) \\ \cong \mathrm{colim}_{i\in I} \lim_{j\in J} \mathrm{Map}_{\mathcal{C}}(D(j), D'(i)) \quad &\cong \mathrm{colim}_{i\in I} \mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_{j\in J} D(j), D'(i)) \end{aligned}$$

□

Example 1.20 (Compact sets)**.** A set is compact if and only if it is finite.

For the "if" part, we first observe that the set $*$ with one object is compact. As finite sets are finite coproducts of points, Corollary 1.19 shows that they are compact.

To see the "only if" part, let S be an infinite set and consider the category I with objects $\{ x_T \mid T \subset S \text{ finite} \}$ and a unique morphism $x_T \to x_{T'}$ whenever T is contained in T'. An easy check shows that I is filtered, and that S is the colimit of the functor $D : I \to \mathrm{Set}$ given by $x_T \mapsto T$. If S were compact, then $\mathrm{Map}_{\mathrm{Set}}(S, S) \cong \mathrm{colim}_{i\in I} \mathrm{Map}_{\mathrm{Set}}(S, D(i))$ and we could factor the identity map $S \to S$ through a finite subset, which is absurd.

Exercise 1.21 (Compact topological spaces)**.** *Compact objects in the category of topological spaces are finite sets with the discrete topology. We will revisit this example later.*

Example 1.22 (Compact modules)**.** A (left) module M over a ring R is compact if and only if it is finitely presented. The proof is almost identical to Example 1.20.

First observe that R is compact because $\mathrm{Map}_R(R, M) \cong M$ and the forgetful functor $\mathrm{Mod}_R \to \mathrm{Set}$ preserves filtered colimits by Exercise 1.17. Since any finitely presented R-module is an iterated finite colimit of copies of R, the "if" part follows.

For the converse direction, we need that any R-module is a filtered colimit of finitely presented modules; we leave this as an exercise. If M is compact, then we can factor the identity map on M through a finitely presented submodule. This shows that M is a summand of a finitely presented module, and hence finitely presented itself.

We have completed the first step towards the desired reformulation of Proposition 1.6:

Corollary 1.23. *A module $Q \in \mathrm{Mod}_R$ is finitely presented if and only if the associated functor $G = \mathrm{Map}_{\mathrm{Mod}_R}(Q, -) : \mathrm{Mod}_R \to \mathrm{Mod}_{\mathbb{Z}}$ preserves filtered colimits.*

Remark 1.24. Since the forgetful functor $\mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Set}$ preserves and reflects filtered colimits, this is an instance of Definition 1.18.

Projectivity. We now give a reformulation of the condition that a module $Q \in \mathrm{Mod}_R$ be projective, with an eye towards later generalisations. First, we recall a well-known result in homological algebra:

Proposition 1.25. *Given a module $Q \in \mathrm{Mod}_R$, the following are equivalent:*

a) Q is a summand of a free module;
b) The functor $\mathrm{Map}_R(Q, -)$ preserves surjections;
c) The functor $\mathrm{Map}_R(Q, -)$ preserves short exact sequences;
d) The functor $\mathrm{Map}_R(Q, -)$ preserves cokernels.

If these conditions hold, we call the module Q projective.

We will reformulate the "cokernel" condition d) using the following notion:

Definition 1.26. A *reflexive pair* in a category $\mathcal{C}$ is a diagram consisting of two arrows $d_0, d_1 : X_1 \rightrightarrows X_0$ and a common section $s : X_0 \to X_1$ satisfying $f \circ s = g \circ s = \mathrm{id}_{X_0}$. In other words, it is a $\Delta^{op}_{\leqslant 1}$-indexed diagram, where $\Delta_{\leqslant 1}$ is the category of nonempty ordered sets of cardinality $\leqslant 1$; we will return to this perspective in the next lectures.

$$\begin{tikzcd} \bullet \arrow[r, bend left, "d_0"] \arrow[r, bend right, "d_1"'] & \bullet \arrow[l, "s"'] \end{tikzcd}$$

A *reflexive coequaliser* is the colimit of a reflexive pair. Note that this agrees with the coequaliser of the arrows d_0 and d_1.

We also record the following notion:

Definition 1.27. A functor $F : \mathrm{Mod}_R \to \mathrm{Mod}_{\mathbb{Z}}$ is called *additive* if for all M, N, the functor $\mathrm{Map}_R(M, N) \to \mathrm{Map}_{\mathbb{Z}}(FM, FN)$ is a homomorphism of abelian groups.

Condition $d)$ in Proposition 1.25 can be reformulated in terms of reflexive coequalisers:

Proposition 1.28. *An additive functor $F : \mathrm{Mod}_R \to \mathrm{Mod}_{\mathbb{Z}}$ preserves cokernels if and only if it preserves reflexive coequalisers.*

Proof. Assume that F preserves cokernels. The coequaliser of a reflexive pair $A \underset{g}{\overset{f}{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}}} B$ is the cokernel of $A \xrightarrow{f-g} B$. As F is additive, this shows that it preserves reflexive coequalisers. Now assume that F preserves reflexive coequalisers. The cokernel of $A \xrightarrow{f} B$ agrees with the coequaliser of the reflexive pair $A \oplus B \underset{\mathrm{id}_B}{\overset{f+\mathrm{id}_B}{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}}} B$, which implies the claim. $\square$

Corollary 1.29. *A module $Q \in \mathrm{Mod}_R$ is projective if and only if the functor $\mathrm{Map}_R(Q, -)$ preserves reflexive coequalisers.*

Exercise 1.30.

a) Prove that the forgetful functor $\mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Set}$ preserves and reflects reflexive coequalisers.
b) Show that this becomes false once we drop the word "reflexive".

Conservativity. Recall that a functor $G : \mathcal{C} \to \mathcal{D}$ is called *conservative* if $f : X \to Y$ is an isomorphism whenever $G(f)$ is one. We can then reformulate condition (2) of Proposition 1.6 by making the following simple observation:

Proposition 1.31. *Any faithful functor $G : \mathrm{Mod}_R \to \mathrm{Mod}_{\mathbb{Z}}$ is conservative. Any conservative functor which preserves coequalisers is faithful.*

Proof. First assume that G is faithful. If $G(f)$ is an isomorphism, then it is both an epi- and a monomorphism. Since G is faithful, this implies that f is both an epi- and a monomorphism, which shows that f is an isomorphism since Mod_R is an abelian category.

Conversely, assume that G is a conservative functor which preserves coequalisers. Note that arrows $f, g : A \to B$ are equal if and only if in the coequaliser diagram $A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} C$, the map h is an isomorphism; this condition is preserved and reflected by the functor G. $\square$

Coming back to Proposition 1.6, we can now rephrase algebraic conditions imposed on Q in terms of categorical conditions on the functor $G = \mathrm{Hom}_R(Q, -)$:

Q is finitely presented	$\leftrightsquigarrow$	G preserves filtered colimits, i.e. Q is compact;
Q is projective	$\leftrightsquigarrow$	G preserves reflexive coequalisers;
Q is a generator	$\rightsquigarrow$	G is conservative.

1.3. **Monads and Adjunctions.** To construct the crucial diagram in Strategy 1.8, we will first use that G admits a left adjoint to construct a monad T on $\mathrm{Mod}_{\mathbb{Z}}$, and then identify T-algebras with S-modules. We briefly review the categorical notions appearing in this sentence.

Monads. Monads provide a way of axiomatising algebraic structures that is convenient for certain abstract arguments. We start with a simple example:

Example 1.32 (Groups)**.** Traditionally, groups are defined as sets X with a binary multiplication $(x, y) \mapsto x \cdot y$, a unary inverse $x \mapsto x^{-1}$, and a unit e satisfying various axioms.

We could also choose a less economical approach, and specify many more operations, e.g.

$$(x_1, x_2, x_3) \mapsto x_1 \cdot x_3^{10} \cdot x_2^{-1}, \qquad (x_1, x_2, x_3, x_4) \mapsto x_1^4 \cdot x_2^2 \cdot x_3 \cdot x_4^{-15}, \qquad \text{etc.} \tag{1}$$

More precisely, consider the endofunctor $T_{\mathrm{Gp}} : \mathrm{Set} \to \mathrm{Set}$ sending a set X to the set of expressions

$$T_{\mathrm{Gp}}(X) \;:=\; \{\; x_1^{a_1} x_2^{a_2} \dots x_k^{a_k} \;\mid\; k \geqslant 0,\; x_i \in X,\; a_k \in \mathbb{Z} - \{0\},\; x_i \neq x_{i+1} \text{ for all } i. \;\}$$

Here the empty word ( ) is considered a valid element of the set $T_{\mathrm{Gp}}(X)$.

In our uneconomical approach to groups, defining all operations as in (1) amounts to specifying a single map $\alpha : T_{\mathrm{Gp}}(X) \to X$ sending a formal expression $x_1^{a_1} x_2^{a_2} \dots x_k^{a_k}$ to the value of the corresponding product $x_1^{a_1} \cdot x_2^{a_2} \cdot \dots \cdot x_k^{a_k}$ in X.

However, not all such maps $\alpha : T_{\mathrm{Gp}}(X) \to X$ define valid group structures on the set X, as we have not yet imposed any of the group axioms. To fix this, we exhibit additional structure on the endofunctor T_{Gp} by specifying the following natural maps for all sets X:

$$\eta_X : X \to T_{\mathrm{Gp}}(X) \qquad\qquad \mu_X : T_{\mathrm{Gp}}(T_{\mathrm{Gp}}(X)) \to T_{\mathrm{Gp}}(X).$$

The first map η_X takes an element $s \in X$ to the corresponding one-letter word in $T_{\mathrm{Gp}}(X)$. The second map μ_X sends a "word of words" $(x_{11}^{a_{11}} \dots x_{1k_1}^{a_{1k_1}})^{b_1} \dots \dots (x_{n1}^{a_{n1}} \dots x_{nk_n}^{a_{nk_n}})^{b_n}$ in $T_{\mathrm{Gp}}(T_{\mathrm{Gp}}(X))$ to the corresponding word in $T_{\mathrm{Gp}}(X)$ given by

$$\underbrace{(x_{11}^{a_{11}} \dots x_{1k_1}^{a_{1k_1}}) \dots (x_{11}^{a_{11}} \dots x_{1k_1}^{a_{1k_1}})}_{b_1} \dots \dots \underbrace{(x_{n1}^{a_{n1}} \dots x_{nk_n}^{a_{nk_n}}) \dots (x_{n1}^{a_{n1}} \dots x_{nk_n}^{a_{nk_n}})}_{b_n}$$

Here, we have implicitly simplified this word by reducing subwords of the form $x^a x^b$ to x^{a+b}.

Exercise 1.33. *The maps η_X and μ_X are natural in X and satisfy the following identities:*

$$\mu_X \circ T_{\mathrm{Gp}}(\mu_X) \cong \mu_X \circ \mu_{T_{\mathrm{Gp}}(X)}, \qquad \mu_X \circ \eta_{T_{\mathrm{Gp}}(X)} = \mathrm{id}_{T_{\mathrm{Gp}}(X)} = \mu_X \circ T_{\mathrm{Gp}}(\eta_X).$$

Using the natural transformations η and μ, we can now formulate a condition for when a map $\alpha : T_{\mathrm{Gp}}(X) \to X$ defines a group structure on X:

Exercise 1.34. *Given a map $\alpha : T_{\mathrm{Gp}}(X) \to X$, the operations $(x, y) \mapsto \alpha(xy)$, $x \mapsto \alpha(x^{-1})$, $e = \alpha(\,)$ define a group structure on X if and only if $\alpha \circ \eta_X = \mathrm{id}_X$ and $\alpha \circ \mu_X = \alpha \circ T_{\mathrm{Gp}}(\alpha)$.*

We therefore obtain a second definition of what a group is, namely a set X together with a map of sets $T_{\mathrm{Gp}}(X) \to X$ satisfying $\alpha \circ \eta_X = \mathrm{id}_X$ and $\alpha \circ \mu_X = \alpha \circ T_{\mathrm{Gp}}(\alpha)$.

Definitions of this kind can also be given for most other algebraic structures of interest (like modules, rings, Lie algebras, ...). We therefore axiomatise this situation:

Definition 1.35 (Monads)**.** A *monad* on a category $\mathcal{C}$ is an associative algebra object in the monoidal category $\mathrm{End}(\mathcal{C})$ of endofunctors (with the composition product $\circ$).

Concretely, this means that a monad is an endofunctor $T : \mathcal{C} \to \mathcal{C}$ equipped with natural transformations $\mathrm{id}_{\mathcal{C}} \to T$ and $\mu : T \circ T \to T$ such that the following two diagrams commute:

$$\begin{tikzcd} T\circ T\circ T \arrow[r, "T(\mu)"] \arrow[d, "\mu_T"'] & T\circ T \arrow[d, "\mu"] \\ T\circ T \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[d, "T(\eta)"'] \arrow[dr, "\mathrm{id}"] & TT \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

Definition 1.36 (Algebras over monads)**.** An *algebra* over a monad T on $\mathcal{C}$ is a T-module object in the $\mathrm{End}(\mathcal{C})$-tensored category $\mathcal{C}$. Concretely, this means that an algebra is a pair $(A \in \mathcal{C}, \alpha : T(A) \to A)$ for which the following two diagrams commute:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\mathrm{id}_A"'] & T(A) \arrow[d, "\alpha"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} T(T(A)) \arrow[r, "T(\alpha)"] \arrow[d, "\mu_A"'] & T(A) \arrow[d, "\alpha"] \\ T(A) \arrow[r, "\alpha"'] & A \end{tikzcd}$$

We write $\mathrm{Alg}_T(\mathcal{C})$ for the category of T-algebras in $\mathcal{C}$.

In Example 1.32, we constructed a monad T_{Gp} acting on Set whose category of algebras $\mathrm{Alg}_{T_{\mathrm{Gp}}}(\mathrm{Set})$ is equivalent to the category of groups. We can construct similar monads for other algebraic structures:

Exercise 1.37.

a) *Define a monad* T_{Ab} *on the category of sets* Set *such that* $\mathrm{Alg}_{T_{\mathrm{Ab}}}(\mathrm{Set})$ *is equivalent to the category* $\mathrm{Ab} = \mathrm{Mod}_{\mathbb{Z}}$ *of abelian groups.*

b) *Define a monad* T_{Ring} *on the category* Ab *such that* $\mathrm{Alg}_{T_{\mathrm{Ring}}}(\mathrm{Ab})$ *is the category of rings.*

c) *Given a ring* R*, define a monad* T_{Ring} *on* Ab *whose category of algebras is equivalent to the category of (left)* R*-modules.*

Adjunctions. In Example 1.32, we have adopted the perspective that the monad T_{Gp} can be used as a tool for defining the notion of a group.

We could also reverse this logic and try to define the monad T_{Gp} assuming that we already know what a group is. To this end, recall the following standard notion from category theory (which we will later generalise to higher categories):

Definition 1.38 (Adjunctions)**.** An adjunction consists of functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ together with natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ (the "unit"), $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$ (the "counit") for which the following diagrams commute:

$$\begin{tikzcd} F \arrow[r, "F(\eta)"] \arrow[dr, "\mathrm{id}_F"'] & FGF \arrow[d, "\epsilon_F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, "\mathrm{id}_G"'] & GFG \arrow[d, "G(\epsilon)"] \\ & G \end{tikzcd}$$

The functor F is called the left adjoint, whereas G is called a right adjoint; we write $F \dashv G$.

Remark 1.39. Fix an adjunction (F, G, η, ϵ) as in Definition 1.38. For any pair of objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, we obtain natural isomorphisms $\mathrm{Map}_{\mathcal{D}}(FX, Y) \cong \mathrm{Map}_{\mathcal{C}}(X, GY)$. Indeed, given $f : FX \to Y$ in $\mathcal{D}$, we attach the map $\overline{f} : X \to GY$ defined by $\overline{f} = Gf \circ \eta_X$. Conversely, to a map $g : X \to GY$, we attach the map $\overline{g} = \epsilon_Y \circ Fg : FX \to Y$. In fact, specifying natural isomorphisms $\mathrm{Map}_{\mathcal{D}}(FX, Y) \cong \mathrm{Map}_{\mathcal{C}}(X, GY)$ leads to an equivalent definition of adjunctions

Example 3.1 (continued)**.** *There is a free-forgetful adjunction* Free : Set $\leftrightarrows$ Gp : Forget *between the category of sets and the category of groups. The right adjoint* Forget *sends a group to its underlying set, and the left adjoint* Free *builds the free group on a given set. The unit* $\eta_X : X \to \mathrm{Forget}(\mathrm{Free}(X))$ *embeds a set* X *into the free group generated by* X*. The counit* $\epsilon_G : \mathrm{Free}(\mathrm{Forget}(G)) \to G$ *takes a formal product* $g_1^{a_1} \ldots g_n^{a_n}$ *in the free group on the set* G *and computes the corresponding product* $g_1^{a_1} \cdot \ldots \cdot g_n^{a_n}$ *in the group* G*.*

We note that the endofunctor $T_{\mathrm{Gp}} : \mathrm{Set} \to \mathrm{Set}$ *defined above is equal to the composite* $\mathrm{Forget} \circ \mathrm{Free}$*. The transformation* $\mathrm{id}_{\mathrm{Set}} \to T_{\mathrm{Gp}}$ *agrees with the unit* η *of the adjunction, and the monad multiplication* $\mu : T_{\mathrm{Gp}} \circ T_{\mathrm{Gp}} \to T_{\mathrm{Gp}}$ *is given by* $G\epsilon_F : GFGF \to GF$*.*

The functor $\mathrm{Gp} \to \mathrm{Alg}_{T_{\mathrm{Gp}}}(\mathrm{Set})$ *sending a group* G *to the* T_{Gp}*-algebra*

$$\left(\mathrm{Forget}(G)\ ,\ T_{\mathrm{Gp}}(\mathrm{Forget}(G)) \xrightarrow{\mathrm{Forget}(\epsilon_G)} \mathrm{Forget}(G)\right)$$

gives the equivalence between groups and T_{Gp}*-algebras mentioned above.*

Indeed, we obtain a monad for every adjunction:

Exercise 1.40 (Monads from adjunctions)**.** *Given an adjunction* $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ *with unit* $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ *and counit* $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$*, show that the endofunctor* $T = GF$ *is equipped with the structure of a monad with unit* $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ *and multiplication* $G\epsilon_F : T \circ T \to T$*.*

Exercise 1.41. *Given a monad* T *on a category* $\mathcal{C}$*, consider the functor* $\mathrm{Free}_T : \mathcal{C} \to \mathrm{Alg}_T(\mathcal{C})$ *sending an object* $X \in \mathcal{C}$ *to the* T*-algebra* $(TX,\ T(T(X)) \xrightarrow{\mu_X} T(X))$*.*

a) Prove that Free_T *is a left adjoint to the forgetful functor* $\mathrm{Forget}_T : \mathrm{Alg}_T(\mathcal{C}) \to \mathcal{C}$*.*

b) Verify that the adjunction $\mathrm{Free}_T \dashv \mathrm{Forget}_T$ *induces the monad* T*.*

This implies the interesting fact that any monad is induced by an adjunction.

Notation 1.42. We will usually denote the free T-algebra on an object $X \in \mathcal{C}$ by $T(X)$ instead of $\mathrm{Free}_T(X)$. Moreover, we will often drop the functor Forget_T from our notation.

If $T = GF$ is a monad obtained from an adjunction $F \dashv G$, we always obtain a functor

$$\widetilde{G} : \mathcal{D} \to \mathrm{Alg}_T(\mathcal{C})$$

sending an object $X \in \mathcal{D}$ to the T-algebra $(G(X)\ ,\ T(G(X)) \xrightarrow{G(\epsilon_X)} G(X))$.

Coming back to Proposition 1.6, we can now give a purely categorical construction of the category Mod_S and the functor $\widetilde{G} : \mathrm{Mod}_R \to \mathrm{Mod}_S$ for $S = \mathrm{End}_R(Q)^{op}$, as desired.

Observation 1.43. *The functor $G = \mathrm{Map}_R(Q, -)$ admits a left adjoint given by $F = Q \otimes (-)$.*

This tensor-hom-adjunction

$$Q \otimes (-) : \mathrm{Mod}_{\mathbb{Z}} \leftrightarrows \mathrm{Mod}_R : \mathrm{Map}_R(Q, -)$$

induces a monad T_Q on $\mathrm{Ab} \cong \mathrm{Mod}_{\mathbb{Z}}$ which sends M to $\mathrm{Map}_R(Q, Q \otimes M)$. Hence $T_Q(\mathbb{Z}) = \mathrm{End}_R(Q)$.

In fact, we can use the conditions on Q to identify the endofunctor T more explicitly.

Observation 1.44. *The functor $G = \mathrm{Map}_R(Q, -)$ preserves biproducts.*

As $G = \mathrm{Map}_R(Q, -)$ also preserves filtered colimits and reflexive coequalisers, it must preserve small colimits. As this is also true for the left adjoint $Q \otimes (-)$, we deduce that the monad $T_Q : \mathrm{Ab} \to \mathrm{Ab}$ preserves small colimits.

Exercise 1.45.

a) Given two rings R_1, R_2, show that a functor $\mathrm{Mod}_{R_1} \to \mathrm{Mod}_{R_2}$ is of the form $M \mapsto B \otimes_{R_1} M$ for some (R_2, R_1)-bimodule B if and only if it is right exact and preserves coproducts (this is known as the Eilenberg-Watts theorem).

b) Identify $\mathrm{Alg}_{T_Q}(\mathrm{Ab})$ with the category of left modules Mod_S over the ring $S = \mathrm{End}_R(Q)^{op}$.

1.4. **The Barr-Beck Theorem.** To prove Proposition 1.6, it remains to show that the induced functor $\widetilde{G} \to \mathrm{Mod}_S \cong \mathrm{Alg}_{T_Q}$ is an equivalence. We will deduce this from the important Barr–Beck theorem, which we will now review. First, let us introduce some terminology:

Definition 1.46. An adjunction $F \dashv G$ with associated monad T is *monadic* if the induced functor $\widetilde{G} : \mathcal{D} \to \mathrm{Alg}_T(\mathcal{C})$ is an equivalence.

In the case of groups, we have seen in Example 1.32 that the forgetful-free adjunction is monadic, thereby giving an alternative definition of groups as T_{Gp}-algebras.

However, not all adjunctions share this desirable property:

Exercise 1.47 (Non-monadic adjunctions)**.** *Consider the adjunction $F : \mathrm{Set} \rightleftarrows \mathrm{Top} : G$ between sets and topological spaces whose right adjoint G sends a space to its underlying set of points, and whose left adjoint F equips a set with the discrete topology.*

Show that this adjunction is not monadic. Hint: what does G do to isomorphisms?

Can you find an example of a conservative non-monadic adjunction?

The Barr-Beck theorem establishes a simple criterion for when an adjunction is monadic:

Theorem 1.48 (Barr-Beck theorem, crude version)**.**
Assume that an adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ satisfies the following two properties:

a) $\mathcal{D}$ admits and G preserves reflexive coequalisers;

b) G is conservative (i.e. reflects isomorphisms).

Then $(F \dashv G)$ is monadic, i.e. $\widetilde{G} : \mathcal{D} \xrightarrow{\cong} \mathrm{Alg}_T(\mathcal{C})$ is an equivalence.

In Definition 1.26, we have introduced the notion of "reflexive coequaliser". To prove Theorem 1.48, we will also need a second notion of coequaliser, which looks similar, but is in fact quite different:

Definition 1.49 (Split coequaliser)**.** Two parallel arrows $d_0, d_1 : X_1 \rightrightarrows X_0$ in a category $\mathcal{C}$ are called a *split pair* if there exist arrows

$$h : X_0 \to X_{-1}, \qquad s : X_{-1} \to X_0, \qquad t : X_0 \to X_1$$

satisfying the following identities:

$$hd_0 = hd_1 \qquad hs = \mathrm{id}_{X_{-1}} \qquad d_0 t = \mathrm{id}_{X_0} \qquad d_1 t = sh$$

$$\begin{tikzcd} \bullet \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \bullet \arrow[l, bend left, "t"] \arrow[r, "h"] & \bullet \arrow[l, bend left, "s"] \end{tikzcd}$$

Exercise 1.50. *Show that in the situation of Definition 1.49, $X_1 \rightrightarrows X_0 \to X_{-1}$ is a coequaliser. Deduce that it is preserved by any functor – we call this an absolute colimit.*

Using split coequalisers, we can build canonical free resolutions of algebras over monads:

Proposition 1.51 (Free resolutions)**.** *Fix a monad T on a category $\mathcal{C}$ and a T-algebra specified by $(A, \alpha : T(A) \to A)$. The following diagram of T-algebras is a coequaliser in $\mathrm{Alg}_T(\mathcal{C})$:*

$$T(T(A)) \underset{\mu_A}{\overset{T(\alpha)}{\rightrightarrows}} T(A) \xrightarrow{\alpha} A \tag{2}$$

Here, we have used the free functor $\mathcal{C} \to \mathrm{Alg}_T(\mathcal{C})$ from Exercise 1.41 (using Notation 1.42), which sends an object $X \in \mathcal{C}$ to the free T-algebra $(T(X),\ T(T(X)) \xrightarrow{\mu_X} T(X))$ on X.

Proof. Observe that after applying the forgetful functor $\mathrm{Alg}_T(\mathcal{C}) \to \mathcal{C}$, the above diagram is part of a split coequaliser with maps $s = \eta_A : A \to T(A)$ and $t = \eta_{T(A)} : T(A) \to T(T(A))$.

To verify that (2) is also a coequaliser in $\mathrm{Alg}_T(\mathcal{C})$, assume we are given a T-algebra $(B, \beta : T(B) \to B)$ together with a map of T-algebras $f : TA \to B$ with $f \circ T(\alpha) = f \circ \mu_A$. By Exercise 1.50, there is a unique $g = f \circ \eta_A$ in $\mathcal{C}$ such that the following triangle commutes:

$$\begin{tikzcd} TA \arrow[r, "\alpha"] \arrow[dr, "f"'] & A \arrow[d, dashed, "g"] \\ & B \end{tikzcd}$$

Hence, it suffices to check that g is a map of T-algebras, which follows from the computation

$$\beta \circ Tf \circ T(\eta_A) = f \circ \mu_A \circ T(\eta_A) = f = f \circ \mu_A \circ \eta_{TA} = f \circ T(\alpha) \circ \eta_{TA} = (f \circ \eta_A) \circ \alpha.$$

We have used that f is a map of T-algebras, the monad axioms for T, and the naturality of η. ☐

With these free resolutions at our disposal, we can now prove the Barr-Beck theorem.

Proof of Theorem 1.48. We proceed in three main steps.

Step 1: Left adjoint $\widetilde{F}$ to $\widetilde{G}$. We have a commuting triangle

$$\begin{tikzcd} & \mathrm{Alg}_T(\mathcal{C}) \arrow[d, "\mathrm{Forget}_T"] \\ \mathcal{D} \arrow[ur, dashed, "\widetilde{G}"] \arrow[r, "G"'] & \mathcal{C} \end{tikzcd}$$

where both G and Forget_T admit left adjoints (cf. Exercise 1.41).

As left adjoints of commuting right adjoints commute, we know that *if* $\widetilde{G}$ admits a left adjoint $\widetilde{F}$, then its value on free T-algebras must be given by $\widetilde{F}(T(X)) = F(X)$.

Since left adjoints also preserve small colimits, Proposition 1.51 motivates us to define the value of $\widetilde{F}$ on a general T-algebra (A, α) as the following coequaliser in $\mathcal{D}$:

$$F(T(A)) \underset{\epsilon_{FA}}{\overset{F(\alpha)}{\rightrightarrows}} F(A) \xrightarrow{\theta} \widetilde{F}(A) \tag{3}$$

This makes sense as $F(T(A)) \underset{\epsilon_{FA}}{\overset{F(\alpha)}{\rightrightarrows}} F(A)$ is a reflexive pair in $\mathcal{D}$ with common section $F\eta_A$. One easily extends this definition to morphisms of T-algebras.

To verify that $\widetilde{F}$ is indeed left adjoint to $\widetilde{G}$, we make the following computation:

$$\begin{array}{l}
\widetilde{F}(A,\alpha) \to B \\ \hline
FA \xrightarrow{f} B \text{ s.t. } f \circ F(\alpha) = f \circ \epsilon_{FA} \\ \hline
A \xrightarrow{\overline{f}} B \text{ s.t. } \overline{f} \circ \alpha = G(\epsilon_B)G(F\overline{f}) \\ \hline
(A,\alpha) \to \widetilde{G}(B) = (GB, G\epsilon_B).
\end{array}$$

In the second step, we have used that $\overline{f} \circ \alpha = \overline{f \circ F(\alpha)} = \overline{f \circ \epsilon_{FA}} \overset{3)}{=} G(f) \overset{4)}{=} G(\epsilon_B)G(F(\overline{f}))$. Here $\overline{(\ )}$ denotes the adjoint bijection on morphisms introduced in Remark 1.39. The first two equalities are straightforward; equalities 3) and 4) follow from the commutative diagrams

$$3)\quad \begin{tikzcd}
GFA \arrow[r] \arrow[d, "\eta_{GFA}"'] \arrow[dr, "\mathrm{id}"] & GB \\
GFGFA \arrow[r, "G\epsilon_{FA}"'] & GFA \arrow[u, "Gf"']
\end{tikzcd}
\qquad\qquad
4)\quad \begin{tikzcd}
FA \arrow[r, "F\overline{f}"] \arrow[dr, "f"'] & FGB \arrow[d, "\epsilon_B"] \\
& B
\end{tikzcd}.$$

Step 2: The unit $\mathrm{id}_{\mathrm{Alg}_T(\mathcal{C})} \to \widetilde{F} \circ \widetilde{G}$ *is an equivalence.*

Given $(A, \alpha) \in \mathrm{Alg}_T(\mathcal{C})$, we have a reflexive coequaliser $F(T(A)) \underset{\epsilon_{FA}}{\overset{F(\alpha)}{\rightrightarrows}} F(A) \xrightarrow{\theta} \widetilde{F}(A)$.

Using that G preserves reflexive coequalisers, we obtain another coequaliser diagram

$$GF(GF(A)) \underset{G\epsilon_{FA}}{\overset{GF(\alpha)}{\rightrightarrows}} GF(A) \xrightarrow{G\theta} G\widetilde{F}(A)$$

As in the proof of *Proposition* 1.51, the following diagram admits a splitting:

$$GF(GF(A)) \underset{G\epsilon_{FA}}{\overset{GF(\alpha)}{\rightrightarrows}} GF(A) \xrightarrow{\alpha} A$$

Having computed the coequaliser of $GF(GF(A)) \underset{G\epsilon_{FA}}{\overset{GF(\alpha)}{\rightrightarrows}} GF(A)$ in two ways, we obtain an isomorphism

$$\begin{tikzcd} GFA \arrow[r, "G\theta"] \arrow[dr, "\alpha"'] & G\widetilde{F}(A,\alpha) \arrow[d, "\cong"] \\ & A \end{tikzcd}$$

We can therefore identify A with $G\widetilde{F}(A,\alpha)$.

Next, we check that $G\epsilon_{\widetilde{F}(A,\alpha)} = \alpha$. Since $\alpha = G\theta$, it suffices to check that $\epsilon_{\widetilde{F}(A,\alpha)} = \theta$. This follows from the following computation:

$$\theta = \theta \circ F\alpha \circ F\eta_A = \theta \circ \epsilon_{FA} \circ F\eta_A = \epsilon_{\widetilde{F}(A,\alpha)} \circ FG(\theta) \circ F\eta_A = \epsilon_{\widetilde{F}(A,\alpha)}$$

In the first and last step, we used the algebra axiom for (A,α), in the second the adjunction axiom relating unit and counit, in the third a naturality square for ϵ.

Altogether, we have verified that $\widetilde{G}(\widetilde{F}(A,\alpha)) = (G\widetilde{F}(A,\alpha), G\epsilon_{\widetilde{F}(A,\alpha)}) \cong (A,\alpha)$.

Step 3: The counit $\widetilde{G} \circ \widetilde{F} \to \mathrm{id}_{\mathcal{D}}$ is an equivalence.
By definition, we have a coequaliser diagram computing $\widetilde{F}(\widetilde{G}(B))$:

$$FGFGB \underset{\epsilon_{FGB}}{\overset{FG\epsilon_B}{\rightrightarrows}} FGB \xrightarrow{\theta} \widetilde{F}(\widetilde{G}(B)) \tag{4}$$

By the universal property, the map $\epsilon_B : FGB \to B$ induces a map $\tau : \widetilde{F}(\widetilde{G}(B)) \to B$.

Applying the functor G to the entire situation, we obtain a diagram

$$\begin{tikzcd} GFGFGB \arrow[r, shift left, "GFG\epsilon_B"] \arrow[r, shift right, "G\epsilon_{FGB}"'] & GFGB \arrow[r] \arrow[dr] & G\widetilde{F}(\widetilde{G}(B)) \arrow[d] \\ & & GB \end{tikzcd}$$

The top line is a coequaliser as G preserves reflexive coequalisers. The diagram

$$GFGFGB \underset{G\epsilon_{FGB}}{\overset{GFG\epsilon_B}{\rightrightarrows}} GFGB \to GB$$

is a split coequaliser (cf. Proposition 1.51). Together, these facts imply that the map $G\widetilde{F}(\widetilde{G}(B)) \to GB$ is an isomorphism, which shows that $\widetilde{F}(\widetilde{G}(B)) \cong B$ as G is conservative. □

We have almost proven a sharper version of the Barr-Beck theorem. To state it, we need a new notion:

Definition 1.52. Given a functor $G : \mathcal{D} \to \mathcal{C}$, a parallel pair $d_0, d_1 : X_1 \rightrightarrows X_0$ is said to be *G-split* if $G(d_0), G(d_1) : X_1 \rightrightarrows X_0$ is a split pair in the sense of Definition 1.49.

We can now state the desired refinement:

Theorem 1.53 (Barr-Beck theorem, precise version)**.**
An adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ is monadic if and only if it has the following two properties:

a) $\mathcal{D}$ admits and G preserves coequalisers of G-split pairs; this means that whenever a pair $d_0, d_1 : X_1 \rightrightarrows X_0$ has the property that $G(X_1), G(X_0) : G(X_1) \rightrightarrows G(X_0)$ is part of a split coequaliser diagram, then $d_0, d_1 : X_1 \rightrightarrows X_0$ admits a colimit, which G preserves.

b) G is conservative (i.e. reflects isomorphisms).

Exercise 1.54. *Taking inspiration from the proof of the crude Barr-Beck Theorem 1.48, prove Theorem 1.53.*

1.5. **Conclusion.** To conclude this lecture, we now give the desired categorical proof of Proposition 1.6. By Observation 1.43, the functor $G = \mathrm{Map}(Q, -) : \mathrm{Mod}_R \to \mathrm{Mod}_{\mathbb{Z}}$ admits a left adjoint $F = Q \otimes (-)$. Writing T_Q for the associated monad on $\mathrm{Mod}_{\mathbb{Z}}$, we obtain a canonical diagram

$$\begin{tikzcd} & \mathrm{Alg}_{T_Q}(\mathrm{Mod}_{\mathbb{Z}}) \arrow[d, "U"] \\ \mathrm{Mod}_R \arrow[ur, dashed, "\widetilde{G}"] \arrow[r, "G"'] & \mathrm{Mod}_{\mathbb{Z}} \end{tikzcd}$$

The functor G preserves biproducts by Observation 1.44, filtered colimits by Corollary 1.23, and reflexive coequalisers by Corollary 1.29. This shows that G and therefore also T_Q preserves small colimits, which allows us to identify $\mathrm{Alg}_{T_Q}(\mathrm{Ab})$ with the category of left $\mathrm{End}_R(Q)^{op}$-modules as in Exercise 1.45. Since G is also conservative by Proposition 1.31, we can apply the crude Barr-Beck theorem Theorem 1.48 to conclude that $\widetilde{G}$ is an equivalence.

Exercise 1.55. *Deduce that all Morita equivalences are realised by the construction in Proposition 1.6, and make this statement precise.*

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