∞ -Categories and Deformation Theory

LECTURE 2. HIGHER CATEGORICAL BACKGROUND

Last week, we discussed the Barr-Beck theorem, which specifies conditions under which an adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ induces an equivalence $\mathcal{D} \xrightarrow{\simeq} \operatorname{Alg}_T(\mathcal{C})$. Here, T = GF is the monad associated with $F \dashv G$, and $\operatorname{Alg}_T(\mathcal{C})$ is the category of T-algebras $(X \in \mathcal{C}, T(X) \xrightarrow{\alpha} X)$.

As a toy application, we proved that if R is a ring and $Q \in \operatorname{Mod}_R^{\heartsuit}$ is a finite projective generator, then R and $S = \operatorname{End}_R(Q)^{op}$ have equivalent categories of (left) modules, via the functor $M \mapsto \operatorname{Map}_R(Q, M)$. In fact, any equivalence of module categories arises in this way.

From Morita to Koszul. The basic setup for Koszul duality is a field k and an augmented associative k-algebra R. Note that this gives k the structure of an R-module.

Taking inspiration from Morita theory, we may ask:

Question. Is the functor $G: \operatorname{Mod}_R \to \operatorname{Mod}_{\operatorname{Hom}_R(k,k)^{op}}, M \mapsto \operatorname{Map}_R(k,M)$ an equivalence?

The answer is a resounding "no", as is manifest from the following simple example:

Example 2.1. For $R = k[\epsilon]/\epsilon^2$, we have $\operatorname{Hom}_R(k,k)^{op} = k$, and the functor G sends $M \in \operatorname{Mod}_{k[\epsilon]/\epsilon^2}$ to $\ker(\epsilon: M \to M)$. Hence G is far from an equivalence.

However, a more sophisticated variant of this construction will give an interesting functor.

Indeed, we can refine the functor $(M \mapsto \operatorname{Map}_R(k, M))$ using homological algebra. If M and N are left modules over a ring R, then $\operatorname{Hom}_R(M, N)$ is only a fragment of a more refined construction called $\mathbb{R} \operatorname{Hom}_R(M, N)$, which is a *complex* of R-modules. It can be computed by chosing a projective resolution $\ldots \to P_2 \to P_1 \to P_0 \to M$ of M and setting

 $\mathbb{R}\operatorname{Hom}_R(M,N) := (\ldots \to 0 \to \operatorname{Hom}_R(P_0,N) \to \operatorname{Hom}_R(P_1,N) \to \ldots)$.

This complex depends on the chosen projective resolution P_{\bullet} , but different resolutions give quasiisomorphic complexes. We will provide a clean formulation of this phenomenon in the language of ∞ -categories in later lectures.

Notation 2.2. We will adjust our notation to stress that chain complexes are henceforth the basic objects of interest. If R is a ring, we will define an ∞ -category whose objects are chain complexes of left R-modules, and we write Mod_R for this enhancement of the classical triangulated category D(R). We will, from now on, write $\operatorname{Mod}_R^{\heartsuit}$ for the ordinary category of ordinary left R-modules.

Notation 2.3. Given a chain complex $M \in Mod_R$, write $\pi_*(M)$ for its homology groups. Concretely, if $M = (\dots \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_{-1} \xrightarrow{d_{-1}} \dots)$, then $\pi_i(M) = \ker(d_i)/\operatorname{im}(d_{i+1})$.

Remark 2.4. Note that $\pi_*(\mathbb{R}\operatorname{Hom}_R(M,N))\cong\operatorname{Ext}_R^{**}(M,N)$ recovers the usual Ext-groups.

We will soon define the structure of a differential graded algebra on $\mathbb{R} \operatorname{Hom}_R(k, k)^{op}$, which will allow us to modify the question raised above:

Question. Is the functor $G: \operatorname{Mod}_R \to \operatorname{Mod}_{\mathbb{R}\operatorname{Hom}_R(k,k)^{op}}, M \mapsto \mathbb{R}\operatorname{Map}_R(k,M)$ an equivalence?

The answer is also "no", but we will see that G sometimes restricts to an interesting equivalence on a subcategory of Mod_R . To formulate and prove this equivalence, we will make use of Lurie's higher categorical generalisation of the classical Barr-Beck theorem. In this lecture, we will introduce the very basics of the theory of ∞ -categories. Due to our time restrictions, we can only scratch the surface – for a more comprehensive treatment, we recommend [Lur09] or the online resource Karedon.

2.1. Simplicial sets. While ∞ -categories might look scary (possibly due to symbol " ∞ "), it is sometimes helpful to remember that they are just simplicial sets satisfying a certain property.

To fix notation, we briefly recall the basic setup of simplicial sets.

Write Δ for the simplex category; its objects are the nonempty finite linearly ordered sets

$$[0] = \{0\}, [1] = \{0 < 1\}, [2] = \{0 < 1 < 2\}, \dots,$$

and morphisms are order-preserving maps.

Definition 2.5 (Simplicial sets). A simplicial set is a functor $\Delta^{op} \rightarrow \text{Set}$. Write **sSet** for the resulting (ordinary) category of simplicial sets.

Notation 2.6 (Simplices and horns). Fix integers $n \ge 0$ and $0 \le i \le n$.

- (1) Write $\Delta^n = \operatorname{Map}_{\Delta}(-, [n]) : \Delta^{op} \to \operatorname{Set}$ for the simplicial set represented by $[n] \in \Delta$.
- (2) Let Λ_i^n be the simplicial set sending $[k] \in \Delta$ to $\{f : [k] \to [n] \text{ s.t. } [n] \setminus \{i\} \notin f([k])\}$.

We refer to Δ^n as the simplicial *n*-simplex, and call Λ^n_i the i^{th} horn of Δ^n .

Definition 2.7 (Mapping objects). Given $X, Y \in \mathbf{sSet}$, define $Y^X \in \mathbf{sSet}$ by

$$(Y^X)_n = \operatorname{Map}_{\mathbf{sSet}}(\Delta^n \times X, Y);$$

the simplicial structure maps are induced by the Yoneda embedding.

The category of simplicial sets is therefore enriched in **sSet**.

Simplicial sets are closely related to Top, the category of (compactly generated) topological spaces. To make this statement precise, we will need, for every $n \ge 0$, the topological *n*-simplex

$$\Delta^{n} = \{ (x_0, \dots, x_n) \in \mathbb{R}^n \mid x_0 + \dots + x_n = 1, \ x_i \ge 0 \}.$$

(5) Any
$$[n] \xrightarrow{f} [m]$$
 induces $\Delta^n \xrightarrow{f_*} \Delta^m$ with $f_*(s_0, \dots, s_n) = (t_0, \dots, t_m), t_j = \sum_{f(i)=j} s_i$.

We can build spaces from simplicial sets:

Definition 2.8 (Geometric realisation). The *geometric realisation* of a simplicial set X is given by $|X| = \operatorname{colim}_{\Delta^n \to X}(\Delta^n)$; this colimit is computed in the ordinary category Top.

We call a simplicial set X weakly contractible if |X| has vanishing homotopy groups.

Exercise 2.9.

- a) Reformulate Definition 2.8 both as a left Kan extension and as a coend.
- b) Give an explicit formula for |X| as a quotient of a coproduct by an equivalence relation.
- c) Describe the spaces $|\Delta^n|$ and $|\Lambda^n_i|$ (cf. Notation 2.6).

We can also go into the reverse direction and attach simplicial sets to spaces:

Definition 2.10 (Singular chains). Given $X \in \text{Top}$, the simplicial set Sing(X) satisfies

$$\operatorname{Sing}(X)_n = \operatorname{Map}_{\operatorname{Top}}(\boldsymbol{\Delta}^n, X),$$

Given a map $[n] \xrightarrow{f} [m]$ in Δ , the corresponding structure map $\operatorname{Sing}(X)_m \to \operatorname{Sing}(X)_n$ is obtained by precomposing with the map $f^* : \Delta^n \xrightarrow{f_*} \Delta^m$ from (5) above. **Exercise 2.11.** Show that the singular chains functor Sing is right adjoint to the geometric realisation functor |-| from Definition 2.8.

Simplicial sets arising as the singular chains of a topological space have a special property:

Definition 2.12 (Kan complexes). A simplicial set X is called a Kan complex if it satisfies the right lifting property for all horns. Concretely, this means that for all n and any $0 \le i \le n$, every map $f_0: \Lambda_i^n \to X$ extends to a map $\overline{f}: \Delta^n \to X$ from the n-simplex:



We can also attach simplicial sets to ordinary categories. For this, we identify the linearly ordered set [n] with the category $(0 \rightarrow 1 \rightarrow ... \rightarrow n)$ and define:

Definition 2.13. The *nerve* of a category C is the simplicial set N(C) with $N(C)_n = Fun_{Cat}([n], C)$; The structure maps are induced by pullback along maps $[n] \rightarrow [m]$ in Δ^{op} .

Remark 2.14. The observant reader might object that if C is not small, then N(C) is too large to be a set. This technical difficulty can be handled rigorously using Grothendieck universes; we refer to [Lur09, Section 1.2.15] for a discussion. In these expository lectures, we will confidently sweep size issues of this kind under the rug.

Simplicial sets which arise as nerves of ordinary categories share a special property:

Exercise 2.15. Show that a simplicial set $X \in \mathbf{sSet}$ is the nerve $N(\mathcal{C})$ of a category \mathcal{C} if and only if for all 0 < i < n and each map $f : \Lambda_i^n \to X$, there is a unique extension to Δ^n .

2.2. Higher categories. To define ∞ -categories, also known as quasi-categories or weak Kan complexes, we relax the uniqueness assertion in Exercise 2.15:

Definition 2.16 (Boardman-Vogt). An ∞ -category is a simplicial set $\mathcal{D} \in \mathbf{sSet}$ such that for all 0 < i < n and each map $f : \Lambda_i^n \to \mathcal{D}$, there is a (not necessarily unique) extension

$$\Lambda^n_i \xrightarrow{f} \mathcal{I}$$

$$\bigwedge^n_i \xrightarrow{f}_{\overline{f}} \mathcal{I}$$

$$\Delta^n \qquad .$$

A functor between ∞ -categories \mathcal{C} and \mathcal{D} is simply a map of simplicial sets.

The 0-simplices of an ∞ -category \mathcal{C} are its *objects*; the 1-simplices are the *morphisms*.

In an ordinary category, we can compose morphisms $x \xrightarrow{f} y, y \xrightarrow{g} z$ and obtain a third morphism $x \xrightarrow{g \circ f} z$. This is reflected in the fact that any $\Lambda_1^2 \to \mathcal{N}(\mathcal{C})$ admits a *unique* filler.

In an ∞ -category \mathcal{D} , the composite of morphisms $x \xrightarrow{f} y$ and $y \xrightarrow{g} z$ is no longer defined uniquely. Instead, there could be many 2-simplices $\Delta^2 \to \mathcal{D}$ of the form





By the inner horn filling condition in Definition 2.16, we can again extend this to a map $\Delta^3 \to \mathcal{D}$, which we think of as an identification between h_1 and h_2 . There could of course be many such 3-simplices, but any two can be "identified" by a 4-simplex, and so on.

Definition 2.17. Given two parallel morphisms $f, g: X \to Y$ in some ∞ -category \mathcal{C} , a homotopy from f to g is a 2-simplex $\Delta^2 \to \mathcal{C}$ such that

$$d_{0}(\sigma) = \operatorname{id}_{Y} \quad d_{1}(\sigma) = g \quad d_{2}(\sigma) = f$$

$$X \xrightarrow{f} Y$$

$$\downarrow_{\operatorname{id}_{Y}}$$

$$V$$

If two morphisms are homotopic, we write $f \simeq g$. Note that this is an equivalence relation, and we write [f] for the set of all morphisms homotopic to f.

Given an ∞ -category, we can define a 1-category by identifying homotopic morphisms:

Definition 2.18 (The homotopy category). The homotopy category hC of an ∞ -category C has objects C_0 . Given two objects $x, y \in hC$, the set of morphisms is given by

$$\operatorname{Map}_{h\mathcal{C}}(x,y) = \{f : x \to y\}/\simeq$$

We define the identity morphism on an object X to be $[id_X]$, and define the composition of $[f]: x \to y$ and $[g]: y \to z$ as $[h]: x \to z$, where $h = d_1(\sigma)$ for any diagram $\sigma: \Delta^2 \to \mathcal{C}$ with $d_2(\sigma) = f$ and $d_0(\sigma) = g$.



Exercise 2.19. Prove that this construction of hC is well-defined and satisfies all the axioms of a category.

Remark 2.20. We will later characterise the homotopy category $h\mathcal{C}$ by a universal property.

We will now specify four important examples of ∞ -categories:

Example 2.21.

- a) For any ordinary category \mathcal{C} , the nerve $N(\mathcal{C})$ is an ∞ -category.
- b) For any given $X \in \text{Top}$, the simplicial set Sing(X) defines an ∞ -category.
- c) The ∞ -category of (compactly generated Hausdorff) spaces is defined in several steps. Let **Kan** \subset **sSet** be the full subcategory spanned by all Kan complexes (cf. Definition 2.12). By Definition 2.7, **Kan** is in fact a simplicial category (i.e. enriched in simplicial sets). For each n, we define a simplicial category Path[n] with objects $0, 1, \ldots, n$, and where $\operatorname{Map}_{\mathfrak{C}[\Delta^n]}(i, j)$ is given by the nerve of the opposite of the poset

$$\{S \mid \{i, j\} \subset S \subset \{i, i+1, i+2, \dots, j-1, j\}\}.$$

We then define the ∞ -category \mathcal{S} of spaces using Cordier's simplicial nerve, i.e. set

$$S_n = \operatorname{Map}_{sCat}(\operatorname{Path}[n], \operatorname{Kan})$$

where sCat is the category of simplicial categories. We leave it as an exercise to define the simplicial structure maps, and to verify that S satisfies the inner horn filling axiom.

d) The ∞ -category $\operatorname{Cat}_{\infty}$ of ∞ -categories is defined by a very similar procedure.

We start with $\operatorname{Cat}_{\infty}^{\Delta}$, the simplicial category whose objects are (small) ∞ -categories and where $\operatorname{Map}_{\operatorname{Cat}_{\infty}^{\Delta}}(\mathcal{C}, \mathcal{D})$ is the largest Kan complex contained in $\mathcal{D}^{\mathcal{C}}$ (cf. Definition 2.7). We then define the ∞ -category $\operatorname{Cat}_{\infty}$ of small ∞ -categories by

$$(\operatorname{Cat}_{\infty})_n = \operatorname{Map}_{\operatorname{sCat}}(\operatorname{Path}[n], \operatorname{Cat}_{\infty}^{\Delta}).$$

Again, we leave the definition of the simplicial structure maps as an exercise.

Note that in $\operatorname{Cat}_{\infty}$, we have not captured noninvertible natural transformations; this would require the theory of $(\infty, 2)$ -categories, which we will not need in this class.

2.3. Colimits. In ordinary category theory, colimits are defined as initial objects in the category of cones over a given diagram. To generalise this definition to ∞ -categories, we will first need to discuss the notion of an initial object. For this, we need to consider the space of maps between two objects in an ∞ -category:

Definition 2.22 (Mapping space). Given objects x, y in an ∞ -category \mathcal{C} , we define the space of right morphisms $\operatorname{Hom}_{\mathcal{C}}^{R}(x, y)$ as the simplicial set with

$$(\operatorname{Hom}_{\mathcal{C}}^{R}(x,y))_{n} = \{z : \Delta^{n+1} \to \mathcal{C} \mid z|_{\Delta^{0,\dots,n}} = \operatorname{id}_{x}, \ z(n+1) = y\}$$

Exercise 2.23. Define the simplicial structure maps in Definition 2.22 and prove that $\operatorname{Hom}_{\mathcal{C}}^{R}(x, y)$ is a Kan complex for all x, y.

We can then define:

Definition 2.24 (Initial objects). An object x in an ∞ -category \mathcal{C} is said to be *initial* if for all $y \in \mathcal{C}$, the simplicial set $\operatorname{Hom}_{\mathcal{C}}^{R}(x, y)$ is weakly contractible.

To define cones in ∞ -categories, we will make use of the following notion:

Definition 2.25 (Join). Given $X, Y \in \mathbf{sSet}$, we define a new simplicial set $X \star Y$ with

$$(X \star Y)_n = X_n \cup \bigcup_{a+b=n-1} X_a \times Y_b \cup Y_n$$

Exercise 2.26.

- a) Complete Definition 2.25 by describing the structure maps of $X \star Y$.
- b) Verify that $\Delta^n \star \Delta^m = \Delta^{n+m-1}$.

Definition 2.27 (∞ -category of cones). Let now $F : I \to C$ be a functor of ∞ -categories. The ∞ -category of cones C_{FI} is given by

$$(\mathcal{C}_{F/})_n = \{\overline{F} : I \star \Delta^n \to \mathcal{C} \mid \overline{F}|_I = F\},\$$

where we again leave the definition of the structure maps as an exercise.

Example 2.28. If $I = (\bullet \leftarrow \bullet \rightarrow \bullet)$ and $F : I \rightarrow C$ picks out a diagram $(b \leftarrow a \rightarrow c)$, then $C_{F/}$ is the ∞ -category of all diagrams



Definition 2.29 (Colimits). Given a diagram $F : I \to C$, a colimit of F is an initial object (cf. Definition 2.24) in the ∞ -category \mathcal{C}_{FI} .

A result of Joyal shows that if a colimit exists, then it is unique up to a contractible space of choices (cf. [Lur09, Proposition 1.2.12.9]). Limits are defined in a dual fashion.

We will often want to talk about filtered colimits in a higher categorical setting. To this end, we generalise the notion of a filtered category from ordinary to higher categories. Given $n \ge 0$, write $\partial \Delta^n$ for the simplicial subset of Λ_{n+1}^{n+1} spanned by all simplices not containing the vertex n + 1.

Definition 2.30. An ∞ -category I is said to be *filtered* if for all integers $n \ge 0$, any map $f : \partial \Delta^n \to I$, extends to Λ_{n+1}^{n+1} :



The case n = 0 shoes that I is nonempty. For n = 1, we conclude that for any diagram $\partial \Delta^1 \rightarrow I$ picking out two objects x, y, we can find an object z and morphisms $x \rightarrow z, y \rightarrow z$.

Exercise 2.31. Show that if I is the nerve of an ordinary category, then Definition 2.30 recovers Definition 1.9.

In the ordinary category Set of sets, we saw in Exercise 1.13 that filtered colimits commuted with finite limits. The higher categorical analogue of this fact is given by the following result:

Proposition 2.32. Filtered colimits and finite limits commute in the ∞ -category S of spaces.

We refer to [Lur09, Proposition 5.3.3.3] for a proof of Proposition 2.32. This result illustrates the general paradigm that S plays the same role for ∞ -categories as Set plays for ordinary categories. We generalise Definition 1.18 to this setting:

Definition 2.33 (Compact objects). An object X in an ∞ -category C is called *compact* if the functor $\operatorname{Map}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathcal{S}$ preserves filtered colimits.

We then have the following generalisation of Corollary 1.19, which follows from Proposition 2.32:

Corollary 2.34. Finite colimits of compact objects in an ∞ -category C are compact.

2.4. Monoidal ∞ -categories. To state Lurie's higher categorical Barr-Beck theorem, we will also need the theory of monads (and their algebras) in this setting, which in turn relies on the theory of monoidal (and tensored) ∞ -categories.

2.5. CoCartesian fibrations. To examine ∞ -categories in families, we will need:

Definition 2.35 (coCartesian lifts). Given a map of simplicial sets $p: \mathcal{C} \to S$ and an edge $f: x \to y$ in S, an edge

$$\widetilde{f}:\widetilde{x}\to\widetilde{y}$$

in C is said to be a *p*-coCartesian lift of f if

a) The edge \tilde{f} lifts f, which means that $p(\tilde{f}) = f$.

b) The map $\mathcal{C}_{\widetilde{f}} \to \mathcal{C}_{\widetilde{x}/} \times_{S_{\pi}/} S_{f/}$ is a trivial Kan fibration of simplicial sets.

Condition b) says that in the diagram below, specifying the upper triangle, an element of $C_{\tilde{f}}$, is equivalent to compatibly specifying $(\tilde{x} \to \tilde{z}) \in C_{\tilde{x}/}$ and the lower triangle, an element of $S_{f/}$.

(6)
$$\begin{array}{c} \widetilde{x} & \overbrace{f}{\widetilde{y}} \widetilde{y} \\ \downarrow & \downarrow \\ x & \overbrace{f}{\widetilde{y}} y \\ z \end{array}$$

Definition 2.36 (CoCartesian fibration). A map $\mathcal{C} \xrightarrow{p} S$ in **sSet** is a *coCartesian fibration* if

(1) p is an *inner fibration*, i.e. it satisfies the right lifting property for all inner horns:



(2) Given $x \xrightarrow{f} y$ in S and $\widetilde{x} \in \mathcal{C}$ with $p(\widetilde{x}) = x$, there is a p-coCartesian lift $\widetilde{x} \xrightarrow{\widetilde{f}} \widetilde{y}$ of f.

As a heuristic, it might be helpful to think of coCartesian fibrations as bundles with flat connection; in this picture, coCartesian lifts correspond to paths along the connection.

2.6. Unstraightening. One can show that coCartesian fibrations over S are equivalent to functors from S into the ∞ -category Cat_{∞}. The proof of this result is challenging, and we refer to [Lur09, Section 3.2] for a more comprehensive treatment.

We will content ourselves with constructing coCartesian fibrations for *certain* functors to Cat_{∞} . More precisely, let J be an ordinary category and fix a functor

$$F: J \to \mathbf{sSet}$$

Definition 2.37 (Relative nerve). The relative nerve $N_F(J)$ is the simplicial set over N(J) with

$$N_{F}(J)_{0} = \{(j_{0} \in N(J)_{0}, x_{0} \in F(j_{0})\} \\ N_{F}(J)_{1} = \{(j_{0} \rightarrow j_{1}) \in N(J)_{1}, x_{1} \in F(j_{1}), F(j_{0} \rightarrow j_{1})(x_{0}) \rightarrow x_{1}\} \\ N_{F}(J)_{2} = \begin{cases} j_{0} \longrightarrow j_{1} & F(j_{0} \rightarrow j_{2})(x_{0}) \rightarrow F(j_{1} \rightarrow j_{2})(x_{1}) \\ \downarrow & x_{1} \in F(j_{1}), F(j_{0} \rightarrow j_{2})(x_{0}) \rightarrow x_{2} \\ x_{2} \in F(j_{2}), F(j_{1} \rightarrow j_{2})(x_{1}) \rightarrow x_{2} \end{cases} \end{cases}$$

Exercise 2.38. a) Write down $N_F(J)_n$ for all n and check that it is a simplicial set. b) Show that if F(j) is an ∞ -category for all j, then $N_F(J) \xrightarrow{p} N(J)$ is a coCartesian fibration.

2.7. Monoidal ∞ -categories. We are finally in a position to define monoidal ∞ -categories.

But first, we observe that the category Δ^{op} admits an alternative description. Indeed, the objects of Δ^{op} can be written as

[0] = (-+), [1] = (-+), [2] = (-++), [3] = (-++), ...

Morphisms from [n] to [m] are maps which preserve the order and send – to – and + to +:



Exercise 2.39. Show that the category defined in this way indeed agrees with the opposite of the usual simplex category Δ .

Informally, we think of the bullets as placeholders of potential elements in a monoidal category. The symbols + and – will act as "bins"; arrows will parametrise multiplications.

We give a name to the morphisms which "throw away" all but one element:

Definition 2.40. Given $n \ge 0$ and $1 \le i \le n$, we write $\rho_i^n : [n] \to [1]$ for the morphism



This motivates the following definition:

Definition 2.41 (Monoidal ∞ -categories). A monoidal ∞ -category is a coCartesian fibration $p: \mathcal{C}^{\otimes} \to \mathcal{N}(\Delta^{op})$ such that for all n, the following morphism is an equivalence:

$$\mathcal{C}^{\circledast}_{[n]} \xrightarrow{\prod_{i=1}^{n} (\rho_{i}^{n})_{!}} \prod_{i=1}^{n} \mathcal{C}^{\circledast}_{[1]} \qquad (\text{Segal condition})$$

Here $\mathcal{C}^{\circledast}_{[n]}$ denotes the fibre of p over [n], and $\mathcal{C}^{\circledast}_{[n]} \xrightarrow{(\rho_i^n)_!} \mathcal{C}^{\circledast}_{[1]}$ is the functor associated with ρ_i^n .

Remark 2.42. The functor $(\rho_i^n)_!$ sends $x \in \mathcal{C}_{[n]}^{\circledast}$ to the endpoint of a coCartesian lift of ρ_i^n starting at \tilde{x} . For a complete definition, we refer to [Lur09, Section 2.2.1].

The monoidal product \circ is determined, up to equivalence, by the following composite:

$$\mathcal{C}^{\circledast}_{[1]} \times \mathcal{C}^{\circledast}_{[1]} \xleftarrow{\simeq} \mathcal{C}^{\circledast}_{[2]} \xrightarrow{m_!} \mathcal{C}^{\circledast}_{[1]},$$

where $m: [2] \rightarrow [1]$ is the morphism in Δ^{op} represented by the diagram



Exercise 2.44. Define the monoidal unit 1 of a monoidal ∞ -category $\mathcal{C}^{\otimes} \to \mathcal{N}(\Delta^{op})$.

Notation 2.45. We will often say "let $(\mathcal{C}, \circ, \mathbf{1})$ be a monoidal ∞ -category" instead of "let $\mathcal{C}^{\circledast} \to \mathcal{N}(\Delta^{op})$ be a monoidal ∞ -category with $\mathcal{C}^{\circledast}_{[1]} \simeq \mathcal{C}$, multiplication \circ , and unit $\mathbf{1}$ ".

Using the relative nerve from Definition 2.37, we can now equip ∞ -categories of endofunctors $\mathcal{C} = \operatorname{End}(\mathcal{D})$ with monoidal structures:

Definition 2.46 (Endomorphism ∞ -categories). Given an ∞ -category \mathcal{D} , we equip

$$\mathcal{C} = \operatorname{End}(\mathcal{D}) \coloneqq \mathcal{D}^{\mathcal{D}}$$

(cf. Definition 2.7) with the structure of a monoidal ∞ -category as follows.

First, use that \mathcal{C} is a simplicial monoid (under composition) to construct a diagram

$$\mathcal{C} \times \mathcal{C} \quad \stackrel{\longrightarrow}{\longleftrightarrow} \quad \mathcal{C} \quad \stackrel{\longrightarrow}{\longleftrightarrow} [0]$$

Second, apply the relative nerve (cf. Definition 2.37) to obtain a coCartesian fibration

$$\operatorname{End}(\mathcal{D})^{\circledast} \to \operatorname{N}(\Delta^{op}).$$

Exercise 2.47. Check that $\operatorname{End}(\mathcal{D})^{\circledast} \to \operatorname{N}(\Delta^{op})$ is a monoidal ∞ -category.

2.8. Algebra objects. To generalise the notion of a monad to a higher categorical context, we first need to define what we mean by an algebra A in a monoidal ∞ -category $(\mathcal{C}, \circ, \mathbf{1})$.

We certainly want to specify a multiplication map $A \circ A \to A$, which, by diagram (6), is equivalent to lifting the morphism $m: [2] \to [1]$ in Δ^{op} drawn below along $p: \mathcal{C}^{\otimes} \to \mathcal{N}(\Delta^{op})$.



We can also specify higher compositions like

$$A \circ A \circ A \xrightarrow{m \circ \mathrm{id}} A \circ A$$

as lifts of corresponding maps in Δ^{op} . One might hope that algebra objects are simply sections of $p: \mathcal{C}^{\circledast} \to \mathcal{N}(\Delta^{op})$.

This is almost true, but we need to make sure that certain dull morphisms have dull lifts:

Definition 2.48 (Inert morphism). A morphism $f : [n] \to [m]$ in Δ^{op} is *inert* if every bullet • in [m] has a unique preimage in [n]:



Definition 2.49 (Algebras). An algebra in a monoidal ∞ -category $p : \mathcal{C}^{\circledast} \to \mathcal{N}(\Delta^{op})$ is a section $s : \mathcal{N}(\Delta^{op}) \to \mathcal{C}^{\circledast}$ of p sending inert morphisms to p-coCartesian morphisms.

Exercise 2.50. Show that if $s : \mathbb{N}(\Delta^{op}) \to \mathcal{C}^{\circledast}$ specifies an algebra, then s([2]) corresponds to the pair (s([1]), s([1])) under the equivalence $\mathcal{C}^{\circledast}_{[2]} \simeq \mathcal{C} \times \mathcal{C}$.

Finally, we can generalise Section 1.3 to the setting of ∞ -categories:

Definition 2.51 (Monads). A monad on an ∞ -category \mathcal{C} is an algebra object in $\operatorname{End}(\mathcal{C})$.

2.9. Algebras over monads. To state the monadicity theorem, we will need to define what we mean by algebras over a monad. We will use the setup of tensored ∞ -categories. Let $\mathcal{C}^{\circledast} \to N(\Delta^{op})$ be a monoidal ∞ -category, written informally as $(\mathcal{C}, \circ, 1)$.

Definition 2.52 (Tensored ∞ -categories). A *C*-tensored ∞ -category is given by a diagram of ∞ -categories $\mathcal{M}^{\circledast} \xrightarrow{q} \mathcal{C}^{\circledast} \xrightarrow{p} N(\Delta^{op})$ satisfying the following conditions:

a) $p \circ q : \mathcal{M}^{\circledast} \to \mathcal{N}(\Delta^{op})$ is a coCartesian fibration;

b) $q: \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes}$ is a categorical fibration sending $(p \circ q)$ -coCartesian to p-coCartesian edges;

c) For all *n*, the inclusion $\{n\} \subset [n]$ induces an equivalence $\mathcal{M}_{[n]}^{\circledast} \xrightarrow{\simeq} \mathcal{C}_{[n]}^{\circledast} \times \mathcal{M}_{\{n\}}^{\circledast}$.

We say that the ∞ -category $\mathcal{M} := \mathcal{M}_{[0]}^{\circledast}$ is equipped with a \mathcal{C} -tensored structure, written \otimes .

Informally, elements of $\mathcal{M}^{\circledast}_{[n]}$ correspond to tuples $(c_1, c_2, \ldots, c_n, m)$ with $c_i \in \mathcal{C}, m \in \mathcal{M}$; we think of the c_i 's as labels of the bullets and m as a label of the +. The $(p \circ q)$ -coCartesian lifts tensor according to the arrows; for example, the coCartesian lift of the morphism



starting at a tuple $(c_1, c_2, c_3, c_4, c_5, c_6, m)$ ends at the tuple $(1, c_2, 1, c_3 \circ c_4 \otimes c_5, c_6 \otimes m)$.

Example 2.53. Any ∞ -category $\mathcal{M} = \mathcal{D}$ is naturally tensored over the monoidal ∞ -category $\mathcal{C} = \operatorname{End}(\mathcal{D})$, where the tensoring evaluates functors on objects.

To formally construct this tensored structure, observe that the simplicial set $\mathcal{M} = \mathcal{D}$ is equipped with an action by the simplicial monoid $\mathcal{C} = \text{End}(\mathcal{D})$.

We obtain the diagram $N(\Delta^{op}) \times \Delta^1 \to \mathbf{sSet}$ drawn below.

. . .

$$\begin{array}{cccc} \mathcal{C} \times \mathcal{C} \times \mathcal{M} & \overleftrightarrow{\longleftrightarrow} & \mathcal{C} \times \mathcal{M} & \overleftrightarrow{\longleftrightarrow} & \mathcal{M} \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ \mathcal{C} \times \mathcal{C} & & & & & & & \mathcal{C} & & & & & \\ \end{array}$$

Exercise. Applying the relative nerve construction to this diagram gives rise to an C = End(D)-tensored structure on $\mathcal{M} = C$.

Let $\mathcal{C}^{\circledast} \xrightarrow{p} \mathcal{N}(\Delta^{op})$ be a monoidal ∞ -category and $\mathcal{M}^{\circledast} \xrightarrow{q} \mathcal{C}^{\circledast} \xrightarrow{p} \mathcal{N}(\Delta^{op})$ be a \mathcal{C} -tensored ∞ -category. Fix an algebra object A in \mathcal{C} , parametrised by a section $s : \mathcal{N}(\Delta^{op}) \to \mathcal{C}^{\circledast}$ of p.

Definition 2.54 (A-Modules). An A-module M in \mathcal{M} consists of a section $s' : \mathbb{N}(\Delta^{op}) \to \mathcal{M}^{\circledast}$ with $q \circ s' = s$ and such that all morphisms drawn below are sent to $(p \circ q)$ -coCartesian edges:



Informally, an A-module is an element $M \in \mathcal{M}$ with a multiplication map $A \otimes M \to M$ which is unital and associative up to coherent homotopy.

Definition 2.55 (Algebras over monads). Given a monad T on an ∞ -category \mathcal{D} , i.e. an algebra object in the monoidal ∞ -category End(\mathcal{D}), a T-algebra is a T-module object in the End(\mathcal{D})-tensored ∞ -category \mathcal{D} .

Remark 2.56. One could argue that *T*-algebras should be called *T*-modules instead, and this notational convention is indeed implemented in [Lur07]. However, we decided against this for increased consistency with the 1-categorical literature on monads.

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