∞-Categories and Deformation Theory

Lecture 2. Higher Categorical Background

Last week, we discussed the Barr-Beck theorem, which specifies conditions under which an adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ induces an equivalence $\mathcal{D} \xrightarrow{\simeq} \mathrm{Alg}_T(\mathcal{C})$. Here, $T = GF$ is the monad associated with $F \dashv G$, and $\mathrm{Alg}_T(\mathcal{C})$ is the category of T-algebras $(X \in \mathcal{C}, T(X) \xrightarrow{\alpha} X)$.

As a toy application, we proved that if R is a ring and $Q \in \mathrm{Mod}_R^{\heartsuit}$ is a finite projective generator, then R and $S = \mathrm{End}_R(Q)^{op}$ have equivalent categories of (left) modules, via the functor $M \mapsto \mathrm{Map}_R(Q, M)$. In fact, any equivalence of module categories arises in this way.

From Morita to Koszul. The basic setup for Koszul duality is a field k and an augmented associative k-algebra R. Note that this gives k the structure of an R-module.

Taking inspiration from Morita theory, we may ask:

Question. Is the functor $G : \mathrm{Mod}_R \to \mathrm{Mod}_{\mathrm{Hom}_R(k,k)^{op}}$, $M \mapsto \mathrm{Map}_R(k, M)$ an equivalence?

The answer is a resounding *"no"*, as is manifest from the following simple example:

Example 2.1. For $R = k[\epsilon]/\epsilon^2$, we have $\mathrm{Hom}_R(k,k)^{op} = k$, and the functor G sends $M \in \mathrm{Mod}_{k[\epsilon]/\epsilon^2}$ to $\ker(\epsilon : M \to M)$. Hence G is far from an equivalence.

However, a more sophisticated variant of this construction will give an interesting functor.

Indeed, we can refine the functor $(M \mapsto \mathrm{Map}_R(k, M))$ using homological algebra. If M and N are left modules over a ring R, then $\mathrm{Hom}_R(M, N)$ is only a fragment of a more refined construction called $\mathbb{R}\mathrm{Hom}_R(M, N)$, which is a *complex* of R-modules. It can be computed by chosing a projective resolution $\ldots \to P_2 \to P_1 \to P_0 \to M$ of M and setting

$$\mathbb{R}\mathrm{Hom}_R(M, N) \quad := \quad (\ldots \to \quad 0 \quad \to \quad \mathrm{Hom}_R(P_0, N) \quad \to \quad \mathrm{Hom}_R(P_1, N) \qquad \to \ldots) \ .$$

This complex depends on the chosen projective resolution $P_\bullet$, but different resolutions give quasi-isomorphic complexes. We will provide a clean formulation of this phenomenon in the language of ∞-categories in later lectures.

Notation 2.2. We will adjust our notation to stress that chain complexes are henceforth the basic objects of interest. If R is a ring, we will define an ∞-category whose objects are chain complexes of left R-modules, and we write Mod_R for this enhancement of the classical triangulated category $D(R)$. We will, from now on, write $\mathrm{Mod}_R^{\heartsuit}$ for the ordinary category of ordinary left R-modules.

Notation 2.3. Given a chain complex $M \in \mathrm{Mod}_R$, write $\pi_*(M)$ for its homology groups. Concretely, if $M = (\ldots \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_{-1} \xrightarrow{d_{-1}} \ldots)$, then $\pi_i(M) = \ker(d_i)/\mathrm{im}(d_{i+1})$.

Remark 2.4. Note that $\pi_*(\mathbb{R}\mathrm{Hom}_R(M, N)) \cong \mathrm{Ext}_R^{-*}(M, N)$ recovers the usual Ext-groups.

We will soon define the structure of a differential graded algebra on $\mathbb{R}\mathrm{Hom}_R(k,k)^{op}$, which will allow us to modify the question raised above:

Question. Is the functor $G : \mathrm{Mod}_R \to \mathrm{Mod}_{\mathbb{R}\mathrm{Hom}_R(k,k)^{op}}$, $M \mapsto \mathbb{R}\mathrm{Map}_R(k, M)$ an equivalence?

The answer is also "no", but we will see that G sometimes restricts to an interesting equivalence on a subcategory of Mod_R. To formulate and prove this equivalence, we will make use of Lurie's higher categorical generalisation of the classical Barr-Beck theorem.

In this lecture, we will introduce the very basics of the theory of ∞-categories. Due to our time restrictions, we can only scratch the surface – for a more comprehensive treatment, we recommend [Lur09] or the online resource Karedon.

2.1. **Simplicial sets.** While ∞-categories might look scary (possibly due to symbol "∞"), it is sometimes helpful to remember that they are just simplicial sets satisfying a certain property.

To fix notation, we briefly recall the basic setup of simplicial sets.

Write Δ for the simplex category; its objects are the nonempty finite linearly ordered sets

$$[0] = \{0\}\ ,\quad [1] = \{0 < 1\}\ ,\quad [2] = \{0 < 1 < 2\}\ ,\quad \dots\ ,$$

and morphisms are order-preserving maps.

Definition 2.5 (Simplicial sets)**.** A *simplicial set* is a functor $\Delta^{op} \to \mathrm{Set}$. Write **sSet** for the resulting (ordinary) category of simplicial sets.

Notation 2.6 (Simplices and horns)**.** Fix integers $n \geq 0$ and $0 \leq i \leq n$.

(1) Write $\Delta^n = \mathrm{Map}_\Delta(-, [n]) : \Delta^{op} \to \mathrm{Set}$ for the simplicial set represented by $[n] \in \Delta$.
(2) Let Λ^n_i be the simplicial set sending $[k] \in \Delta$ to $\{f : [k] \to [n] \text{ s.t. } [n]\backslash\{i\} \notin f([k])\}$.

We refer to Δ^n as the simplicial n-simplex, and call Λ^n_i the i^{th} horn of Δ^n.

Definition 2.7 (Mapping objects)**.** Given $X, Y \in \mathbf{sSet}$, define $Y^X \in \mathbf{sSet}$ by

$$(Y^X)_n = \mathrm{Map}_{\mathbf{sSet}}(\Delta^n \times X, Y);$$

the simplicial structure maps are induced by the Yoneda embedding.

The category of simplicial sets is therefore enriched in **sSet**.

Simplicial sets are closely related to Top, the category of (compactly generated) topological spaces. To make this statement precise, we will need, for every $n \geq 0$, the topological n-simplex

$$\mathbf{\Delta}^n = \{(x_0, \dots, x_n) \in \mathbb{R}^n \mid x_0 + \dots + x_n = 1,\ x_i \geq 0\}.$$

$$\text{(1)} \qquad \text{Any } [n] \xrightarrow{f} [m] \text{ induces } \mathbf{\Delta}^n \xrightarrow{f_*} \mathbf{\Delta}^m \text{ with } f_*(s_0, \dots, s_n) = (t_0, \dots, t_m),\ t_j = \sum_{f(i)=j} s_i.$$

We can build spaces from simplicial sets:

Definition 2.8 (Geometric realisation)**.** The *geometric realisation* of a simplicial set X is given by $|X| = \mathrm{colim}_{\Delta^n \to X}(\mathbf{\Delta}^n)$; this colimit is computed in the ordinary category Top.

We call a simplicial set X weakly contractible if $|X|$ has vanishing homotopy groups.

Exercise 2.9.
a) Reformulate Definition 2.8 both as a left Kan extension and as a coend.
b) Give an explicit formula for $|X|$ as a quotient of a coproduct by an equivalence relation.
c) Describe the spaces $|\Delta^n|$ and $|\Lambda^n_i|$ (cf. Notation 2.6).

We can also go into the reverse direction and attach simplicial sets to spaces:

Definition 2.10 (Singular chains)**.** Given $X \in \mathrm{Top}$, the simplicial set $\mathrm{Sing}(X)$ satisfies

$$\mathrm{Sing}(X)_n = \mathrm{Map}_{\mathrm{Top}}(\mathbf{\Delta}^n, X),$$

Given a map $[n] \xrightarrow{f} [m]$ in Δ, the corresponding structure map $\mathrm{Sing}(X)_m \to \mathrm{Sing}(X)_n$ is obtained by precomposing with the map $f^* : \mathbf{\Delta}^n \xrightarrow{f_*} \mathbf{\Delta}^m$ from (1) above.

Exercise 2.11. *Show that the singular chains functor* Sing *is right adjoint to the geometric realisation functor* $|-|$ *from Definition 2.8.*

Simplicial sets arising as the singular chains of a topological space have a special property:

Definition 2.12 (Kan complexes)**.** A simplicial set X is called a *Kan complex* if it satisfies the right lifting property for all horns. Concretely, this means that for all n and any $0 \le i \le n$, every map $f_0 : \Lambda^n_i \to X$ extends to a map $\overline{f} : \Delta^n \to X$ from the n-simplex:

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

We can also attach simplicial sets to ordinary categories. For this, we identify the linearly ordered set $[n]$ with the category $(0 \to 1 \to \ldots \to n)$ and define:

Definition 2.13. The *nerve* of a category $\mathcal{C}$ is the simplicial set $\mathrm{N}(\mathcal{C})$ with $\mathrm{N}(\mathcal{C})_n = \mathrm{Fun}_{\mathrm{Cat}}([n], \mathcal{C})$; The structure maps are induced by pullback along maps $[n] \to [m]$ in Δ^{op}.

Remark 2.14. The observant reader might object that if $\mathcal{C}$ is not small, then $\mathrm{N}(\mathcal{C})$ is too large to be a set. This technical difficulty can be handled rigorously using Grothendieck universes; we refer to [Lur09, Section 1.2.15] for a discussion. In these expository lectures, we will confidently sweep size issues of this kind under the rug.

Simplicial sets which arise as nerves of ordinary categories share a special property:

Exercise 2.15. *Show that a simplicial set* $X \in \mathbf{sSet}$ *is the nerve* $\mathrm{N}(\mathcal{C})$ *of a category* $\mathcal{C}$ *if and only if for all* $0 < i < n$ *and each map* $f : \Lambda^n_i \to X$*, there is a unique extension to* Δ^n*.*

2.2. **Higher categories.** To define ∞-categories, also known as quasi-categories or weak Kan complexes, we relax the uniqueness assertion in Exercise 2.15:

Definition 2.16 (Boardman-Vogt)**.** An ∞-category is a simplicial set $\mathcal{D} \in \mathbf{sSet}$ such that for all $0 < i < n$ and each map $f : \Lambda^n_i \to \mathcal{D}$, there is a (not necessarily unique) extension

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & \mathcal{D} \\ \Delta^n \arrow[ur, dashed, "\overline{f}"'] & . \end{tikzcd}$$

A functor between ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is simply a map of simplicial sets.

The 0-simplices of an ∞-category $\mathcal{C}$ are its *objects*; the 1-simplices are the *morphisms*.

In an ordinary category, we can compose morphisms $x \xrightarrow{f} y$, $y \xrightarrow{g} z$ and obtain a third morphism $x \xrightarrow{g \circ f} z$. This is reflected in the fact that any $\Lambda^2_1 \to \mathrm{N}(\mathcal{C})$ admits a *unique* filler.

In an ∞-category $\mathcal{D}$, the composite of morphisms $x \xrightarrow{f} y$ and $y \xrightarrow{g} z$ is no longer defined uniquely. Instead, there could be many 2-simplices $\Delta^2 \to \mathcal{D}$ of the form

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[dr, dashed] & y \arrow[d, "g"] \\ & z \end{tikzcd}$$

For any two such fillers $\Delta^2 \to \mathcal{D}$ with $\{0,2\}$-edges $h_1, h_2 \in \mathcal{D}_1$, which we think of as two composites of f and g, we obtain a morphism $\Lambda^3_1 \to \mathcal{D}$ depicted below:

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[drr, "h_1"] \arrow[ddrr, "h_2"'] & y \arrow[dr, "g"] \arrow[ddr, "g"'] & \\
& & z \arrow[d, "\mathrm{id}_z"] \\
& & z
\end{tikzcd}$$

By the inner horn filling condition in Definition 2.16, we can again extend this to a map $\Delta^3 \to \mathcal{D}$, which we think of as an identification between h_1 and h_2. There could of course be many such 3-simplices, but any two can be "identified" by a 4-simplex, and so on.

Definition 2.17. Given two parallel morphisms $f, g : X \to Y$ in some ∞-category $\mathcal{C}$, a homotopy from f to g is a 2-simplex $\Delta^2 \to \mathcal{C}$ such that

$$d_0(\sigma) = \mathrm{id}_Y \quad d_1(\sigma) = g \quad d_2(\sigma) = f$$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, "\mathrm{id}_Y"] \\
& Y
\end{tikzcd}$$

If two morphisms are homotopic, we write $f \simeq g$. Note that this is an equivalence relation, and we write $[f]$ for the set of all morphisms homotopic to f.

Given an ∞-category, we can define a 1-category by identifying homotopic morphisms:

Definition 2.18 (The homotopy category)**.** The *homotopy category* $h\mathcal{C}$ of an ∞-category $\mathcal{C}$ has objects $\mathcal{C}_0$. Given two objects $x, y \in h\mathcal{C}$, the set of morphisms is given by

$$\mathrm{Map}_{h\mathcal{C}}(x, y) = \{f : x \to y\}/\simeq.$$

We define the identity morphism on an object X to be $[id_X]$, and define the composition of $[f] : x \to y$ and $[g] : y \to z$ as $[h] : x \to z$, where $h = d_1(\sigma)$ for any diagram $\sigma : \Delta^2 \to \mathcal{C}$ with $d_2(\sigma) = f$ and $d_0(\sigma) = g$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\
& Y
\end{tikzcd}$$

Exercise 2.19. *Prove that this construction of $h\mathcal{C}$ is well-defined and satisfies all the axioms of a category.*

Remark 2.20. We will later characterise the homotopy category $h\mathcal{C}$ by a universal property.

We will now specify four important examples of ∞-categories:

Example 2.21.

a) For any ordinary category $\mathcal{C}$, the nerve $\mathrm{N}(\mathcal{C})$ is an ∞-category.
b) For any given $X \in \mathrm{Top}$, the simplicial set $\mathrm{Sing}(X)$ defines an ∞-category.
c) The ∞-category of (compactly generated Hausdorff) spaces is defined in several steps. Let $\mathbf{Kan} \subset \mathbf{sSet}$ be the full subcategory spanned by all Kan complexes (cf. Definition 2.12). By Definition 2.7, $\mathbf{Kan}$ is in fact a simplicial category (i.e. enriched in simplicial sets). For each n, we define a simplicial category $\mathrm{Path}[n]$ with objects $0, 1, \ldots, n$, and where $\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i,j)$ is given by the nerve of the opposite of the poset
$$\{S \mid \{i,j\} \subset S \subset \{i, i+1, i+2, \ldots, j-1, j\}\}.$$
We then define the ∞-category $\mathcal{S}$ of spaces using Cordier's simplicial nerve, i.e. set
$$\mathcal{S}_n = \mathrm{Map}_{\mathrm{sCat}}(\mathrm{Path}[n], \mathbf{Kan}),$$
where sCat is the category of simplicial categories. We leave it as an exercise to define the simplicial structure maps, and to verify that $\mathcal{S}$ satisfies the inner horn filling axiom.
d) The ∞-category Cat_∞ of ∞-categories is defined by a very similar procedure.
We start with $\mathbf{Cat}_\infty^\Delta$, the simplicial category whose objects are (small) ∞-categories and where $\mathrm{Map}_{\mathrm{Cat}_\infty^\Delta}(\mathcal{C}, \mathcal{D})$ is the largest Kan complex contained in $\mathcal{D}^{\mathcal{C}}$ (cf. Definition 2.7). We then define the ∞-category Cat_∞ of small ∞-categories by
$$(\mathrm{Cat}_\infty)_n = \mathrm{Map}_{\mathrm{sCat}}(\mathrm{Path}[n], \mathbf{Cat}_\infty^\Delta).$$
Again, we leave the definition of the simplicial structure maps as an exercise.
Note that in Cat_∞, we have not captured noninvertible natural transformations; this would require the theory of $(\infty, 2)$-categories, which we will not need in this class.

2.3. **Colimits.** In ordinary category theory, colimits are defined as initial objects in the category of cones over a given diagram. To generalise this definition to ∞-categories, we will first need to discuss the notion of an initial object. For this, we need to consider the space of maps between two objects in an ∞-category:

Definition 2.22 (Mapping space)**.** Given objects x, y in an ∞-category $\mathcal{C}$, we define the *space of right morphisms* $\mathrm{Hom}^R_{\mathcal{C}}(x,y)$ as the simplicial set with
$$(\mathrm{Hom}^R_{\mathcal{C}}(x,y))_n = \{z : \Delta^{n+1} \to \mathcal{C} \mid z|_{\Delta^{0,\ldots,n}} = \mathrm{id}_x,\ z(n+1) = y\}$$

Exercise 2.23. *Define the simplicial structure maps in Definition 2.22 and prove that* $\mathrm{Hom}^R_{\mathcal{C}}(x,y)$ *is a Kan complex for all* x, y*.*

We can then define:

Definition 2.24 (Initial objects)**.** An object x in an ∞-category $\mathcal{C}$ is said to be *initial* if for all $y \in \mathcal{C}$, the simplicial set $\mathrm{Hom}^R_{\mathcal{C}}(x,y)$ is weakly contractible.

To define cones in ∞-categories, we will make use of the following notion:

Definition 2.25 (Join)**.** Given $X, Y \in \mathbf{sSet}$, we define a new simplicial set $X \star Y$ with
$$(X \star Y)_n = X_n \cup \bigcup_{a+b=n-1} X_a \times Y_b \ \cup \ Y_n$$

Exercise 2.26.
a) Complete Definition 2.25 by describing the structure maps of $X \star Y$.
b) Verify that $\Delta^n \star \Delta^m = \Delta^{n+m+1}$.

Definition 2.27 (∞-category of cones)**.** Let now $F : I \to \mathcal{C}$ be a functor of ∞-categories. The ∞*-category of cones* $\mathcal{C}_{F/}$ is given by

$$(\mathcal{C}_{F/})_n = \{\overline{F} : I \star \Delta^n \to \mathcal{C} \mid \overline{F}|_I = F\},$$

where we again leave the definition of the structure maps as an exercise.

Example 2.28. If $I = (\bullet \leftarrow \bullet \rightarrow \bullet)$ and $F : I \to \mathcal{C}$ picks out a diagram $(b \leftarrow a \rightarrow c)$, then $\mathcal{C}_{F/}$ is the ∞-category of all diagrams

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d, dashed] \\ c \arrow[r, dashed] & d \end{tikzcd}$$

Definition 2.29 (Colimits)**.** Given a diagram $F : I \to \mathcal{C}$, a colimit of F is an initial object (cf. Definition 2.24) in the ∞-category $\mathcal{C}_{F/}$.

A result of Joyal shows that if a colimit exists, then it is unique up to a contractible space of choices (cf. [Lur09, Proposition 1.2.12.9]). Limits are defined in a dual fashion.

We will often want to talk about filtered colimits in a higher categorical setting. To this end, we generalise the notion of a filtered category from ordinary to higher categories. Given $n \geq 0$, write $\partial\Delta^n$ for the simplicial subset of Λ^{n+1}_{n+1} spanned by all simplices not containing the vertex $n+1$.

Definition 2.30. An ∞-category I is said to be *filtered* if for all integers $n \geq 0$, any map $f : \partial\Delta^n \to I$, extends to Λ^{n+1}_{n+1}:

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "f"] \arrow[d, hook] & \mathcal{D} \\ \Lambda^{n+1}_{n+1} \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd} .$$

The case $n = 0$ shoes that I is nonempty. For $n = 1$, we conclude that for any diagram $\partial\Delta^1 \to I$ picking out two objects x, y, we can find an object z and morphisms $x \to z$, $y \to z$.

Exercise 2.31. *Show that if I is the nerve of an ordinary category, then Definition 2.30 recovers Definition 1.9 from Lecture 1.*

In the ordinary category Set of sets, filtered colimits commuted with finite limits. The higher categorical analogue of this fact is given by the following result:

Proposition 2.32. *Filtered colimits and finite limits commute in the ∞-category $\mathcal{S}$ of spaces.*

We refer to [Lur09, Proposition 5.3.3.3] for a proof of Proposition 2.32. This result illustrates the general paradigm that $\mathcal{S}$ plays the same role for ∞-categories as Set plays for ordinary categories. We generalise Definition 1.18 from Lecture 1 to this setting:

Definition 2.33 (Compact objects)**.** An object X in an ∞-category $\mathcal{C}$ is called *compact* if the functor $\mathrm{Map}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathcal{S}$ preserves filtered colimits.

We then have the following generalisation of Corollary 1.19 from Lecture 1, which follows from Proposition 2.32:

Corollary 2.34. *Finite colimits of compact objects in an ∞-category $\mathcal{C}$ are compact.*

2.4. **Monoidal ∞-categories.** To state Lurie's higher categorical Barr-Beck theorem, we will also need the theory of monads (and their algebras) in this setting, which in turn relies on the theory of monoidal (and tensored) ∞-categories.

2.5. **CoCartesian fibrations.** To examine ∞-categories in families, we will need:

Definition 2.35 (coCartesian lifts)**.** Given a map of simplicial sets $p : \mathcal{C} \to S$ and an edge $f : x \to y$ in S, an edge

$$\widetilde{f} : \widetilde{x} \to \widetilde{y}$$

in $\mathcal{C}$ is said to be a *p-coCartesian lift* of f if

a) The edge $\widetilde{f}$ lifts f, which means that $p(\widetilde{f}) = f$.
b) The map $\mathcal{C}_{\widetilde{f}} \to \mathcal{C}_{\widetilde{x}/} \times_{S_x/} S_{f/}$ is a trivial Kan fibration of simplicial sets.

Condition b) says that in the diagram below, specifying the upper triangle, an element of $\mathcal{C}_{\widetilde{f}}$, is equivalent to compatibly specifying $(\widetilde{x} \to \widetilde{z}) \in \mathcal{C}_{\widetilde{x}/}$ and the lower triangle, an element of $S_{f/}$.

(2)
$$\begin{tikzcd}
\widetilde{x} \arrow[r, "\widetilde{f}"] \arrow[drr] \arrow[dd, mapsto] & \widetilde{y} \arrow[dr, dashed] \arrow[dd, mapsto] & \\
& & \widetilde{z} \arrow[dd, mapsto] \\
x \arrow[r, "f"] \arrow[drr] & y \arrow[dr] & \\
& & z
\end{tikzcd}$$

Definition 2.36 (CoCartesian fibration)**.** A map $\mathcal{C} \xrightarrow{p} S$ in **sSet** is a *coCartesian fibration* if

(1) p is an *inner fibration*, i.e. it satisfies the right lifting property for all inner horns:

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

(2) Given $x \xrightarrow{f} y$ in S and $\widetilde{x} \in \mathcal{C}$ with $p(\widetilde{x}) = x$, there is a p-coCartesian lift $\widetilde{x} \xrightarrow{\widetilde{f}} \widetilde{y}$ of f.

As a heuristic, it might be helpful to think of coCartesian fibrations as bundles with flat connection; in this picture, coCartesian lifts correspond to paths along the connection.

2.6. **Unstraightening.** One can show that coCartesian fibrations over S are equivalent to functors from S into the ∞-category Cat_∞. The proof of this result is challenging, and we refer to [Lur09, Section 3.2] for a more comprehensive treatment.

We will content ourselves with constructing coCartesian fibrations for *certain* functors to Cat_∞. More precisely, let J be an ordinary category and fix a functor

$$F : J \to \mathbf{sSet}\,.$$

Definition 2.37 (Relative nerve)**.** The *relative nerve* $\mathrm{N}_F(J)$ is the simplicial set over $\mathrm{N}(J)$ with

$$\mathrm{N}_F(J)_0 = \{(j_0 \in \mathrm{N}(J)_0\ ,\ x_0 \in F(j_0))\}$$

$$\mathrm{N}_F(J)_1 = \{(j_0 \to j_1) \in \mathrm{N}(J)_1\ ,\ {\scriptstyle x_0 \in F(j_0) \atop x_1 \in F(j_1)}\ ,\ F(j_0 \to j_1)(x_0) \to x_1\}$$

$$\mathrm{N}_F(J)_2 = \left\{ \begin{tikzcd} j_0 \arrow[r] \arrow[dr] & j_1 \arrow[d] \\ & j_2 \end{tikzcd}\ ,\ {\scriptstyle x_0 \in F(j_0) \atop {x_1 \in F(j_1) \atop x_2 \in F(j_2)}}\ ,\ {\scriptstyle F(j_0 \to j_1)(x_0) \to x_1 \atop {F(j_0 \to j_2)(x_0) \to x_2 \atop F(j_1 \to j_2)(x_1) \to x_2}}\ ,\ \begin{tikzcd} F(j_0 \to j_2)(x_0) \arrow[r] \arrow[dr] & F(j_1 \to j_2)(x_1) \arrow[d] \\ & x_2 \end{tikzcd} \right\}$$

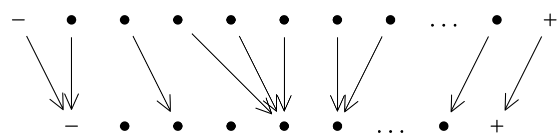
Exercise 2.38. *a) Write down $N_F(J)_n$ for all n and check that it is a simplicial set.*
b) Show that if $F(j)$ is an ∞-category for all j, then $\mathrm{N}_F(J) \xrightarrow{p} \mathrm{N}(J)$ is a coCartesian fibration.

2.7. **Monoidal ∞-categories.** We are finally in a position to define monoidal ∞-categories.

But first, we observe that the category Δ^{op} admits an alternative description. Indeed, the objects of Δ^{op} can be written as

$$[0] = (-\ \ +),\quad [1] = (-\ \bullet\ +),\quad [2] = (-\ \bullet\ \bullet\ +),\quad [3] = (-\ \bullet\ \bullet\ \bullet\ +),\quad \ldots$$

Morphisms from $[n]$ to $[m]$ are maps which preserve the order and send $-$ to $-$ and $+$ to $+$:

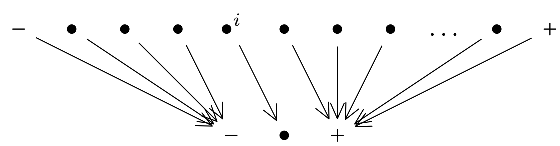


Exercise 2.39. *Show that the category defined in this way indeed agrees with the opposite of the usual simplex category Δ.*

Informally, we think of the bullets as placeholders of potential elements in a monoidal category. The symbols + and − will act as "bins"; arrows will parametrise multiplications.

We give a name to the morphisms which "throw away" all but one element:

Definition 2.40. Given $n \geq 0$ and $1 \leq i \leq n$, we write $\rho_i^n : [n] \to [1]$ for the morphism



This motivates the following definition:

Definition 2.41 (Monoidal ∞-categories)**.** A monoidal ∞-category is a coCartesian fibration $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\Delta^{op})$ such that for all n, the following morphism is an equivalence:

$$\mathcal{C}^{\otimes}_{[n]} \xrightarrow{\ \prod_{i=1}^{n} (\rho_i^n)_!\ } \prod_{i=1}^{n} \mathcal{C}^{\otimes}_{[1]} \qquad \text{(Segal condition)}$$

Here $\mathcal{C}^{\otimes}_{[n]}$ denotes the fibre of p over $[n]$, and $\mathcal{C}^{\otimes}_{[n]} \xrightarrow{(\rho_i^n)_!} \mathcal{C}^{\otimes}_{[1]}$ is the functor associated with ρ_i^n.

Remark 2.42. The functor $(\rho_i^n)_!$ sends $x \in \mathcal{C}^{\otimes}_{[n]}$ to the endpoint of a coCartesian lift of ρ_i^n starting at $\tilde{x}$. For a complete definition, we refer to [Lur09, Section 2.2.1].

Notation 2.43. Informally, we simply say that $\mathcal{C} \simeq \mathcal{C}^{\circledast}_{[1]}$ is equipped with a monoidal structure.

The monoidal product $\circ$ is determined, up to equivalence, by the following composite:

$$\mathcal{C}^{\circledast}_{[1]} \times \mathcal{C}^{\circledast}_{[1]} \xleftarrow{\simeq} \mathcal{C}^{\circledast}_{[2]} \xrightarrow{m_!} \mathcal{C}^{\circledast}_{[1]},$$

where $m : [2] \to [1]$ is the morphism in Δ^{op} represented by the diagram

$$\begin{tikzcd}
- \arrow[dr] & \bullet \arrow[dr] & & \bullet \arrow[dl] & + \arrow[dl] \\
& - & \bullet & + .
\end{tikzcd}$$

Exercise 2.44. *Define the monoidal unit* $\mathbf{1}$ *of a monoidal* ∞*-category* $\mathcal{C}^{\circledast} \to \mathrm{N}(\Delta^{op})$.

Notation 2.45. We will often say "let $(\mathcal{C}, \circ, \mathbf{1})$ be a monoidal ∞-category" instead of "let $\mathcal{C}^{\circledast} \to \mathrm{N}(\Delta^{op})$ be a monoidal ∞-category with $\mathcal{C}^{\circledast}_{[1]} \simeq \mathcal{C}$, multiplication $\circ$, and unit $\mathbf{1}$".

Using the relative nerve from Definition 2.37, we can now equip ∞-categories of endofunctors $\mathcal{C} = \mathrm{End}(\mathcal{D})$ with monoidal structures:

Definition 2.46 (Endomorphism ∞-categories)**.** Given an ∞-category $\mathcal{D}$, we equip

$$\mathcal{C} = \mathrm{End}(\mathcal{D}) \coloneqq \mathcal{D}^{\mathcal{D}}$$

(cf. Definition 2.7) with the structure of a monoidal ∞-category as follows.

First, use that $\mathcal{C}$ is a simplicial monoid (under composition) to construct a diagram

$$\begin{tikzcd}
\dots & \mathcal{C} \times \mathcal{C} \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & \mathcal{C} \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[r, shift left=2] \arrow[r, shift right=2] & {[0]} \arrow[l]
\end{tikzcd}$$

Second, apply the relative nerve (cf. Definition 2.37) to obtain a coCartesian fibration

$$\mathrm{End}(\mathcal{D})^{\circledast} \to \mathrm{N}(\Delta^{op}).$$

Exercise 2.47. *Check that* $\mathrm{End}(\mathcal{D})^{\circledast} \to \mathrm{N}(\Delta^{op})$ *is a monoidal* ∞*-category.*

2.8. **Algebra objects.** To generalise the notion of a monad to a higher categorical context, we first need to define what we mean by an algebra A in a monoidal ∞-category $(\mathcal{C}, \circ, \mathbf{1})$.

We certainly want to specify a multiplication map $A \circ A \to A$, which, by diagram (2), is equivalent to lifting the morphism $m : [2] \to [1]$ in Δ^{op} drawn below along $p : \mathcal{C}^{\circledast} \to \mathrm{N}(\Delta^{op})$.

$$\begin{tikzcd}
- \arrow[dr] & \bullet \arrow[dr] & & \bullet \arrow[dl] & + \arrow[dl] \\
& - & \bullet & + .
\end{tikzcd}$$

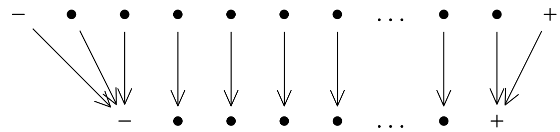
We can also specify higher compositions like

$$A \circ A \circ A \xrightarrow{m \circ \mathrm{id}} A \circ A$$

as lifts of corresponding maps in Δ^{op}. One might hope that algebra objects are simply sections of $p : \mathcal{C}^{\circledast} \to \mathrm{N}(\Delta^{op})$.

This is almost true, but we need to make sure that certain dull morphisms have dull lifts:

Definition 2.48 (Inert morphism)**.** A morphism $f : [n] \to [m]$ in Δ^{op} is *inert* if every bullet $\bullet$ in $[m]$ has a unique preimage in $[n]$:



Definition 2.49 (Algebras)**.** An *algebra* in a monoidal ∞-category $p : \mathcal{C}^{\circledast} \to \mathrm{N}(\Delta^{op})$ is a section $s : \mathrm{N}(\Delta^{op}) \to \mathcal{C}^{\circledast}$ of p sending inert morphisms to p-coCartesian morphisms.

Exercise 2.50. *Show that if $s : \mathrm{N}(\Delta^{op}) \to \mathcal{C}^{\circledast}$ specifies an algebra, then $s([2])$ corresponds to the pair $(s([1]), s([1]))$ under the equivalence $\mathcal{C}^{\circledast}_{[2]} \simeq \mathcal{C} \times \mathcal{C}$.*

Finally, we can generalise Definition 1.35 from Lecture 1 to the setting of ∞-categories:

Definition 2.51 (Monads)**.** A monad on an ∞-category $\mathcal{C}$ is an algebra object in $\mathrm{End}(\mathcal{C})$.

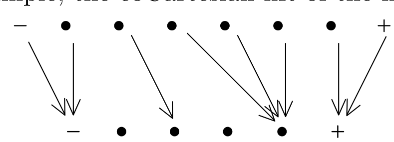
2.9. **Algebras over monads.** To state the monadicity theorem, we will need to define what we mean by algebras over a monad. We will use the setup of tensored ∞-categories. Let $\mathcal{C}^{\circledast} \to \mathrm{N}(\Delta^{op})$ be a monoidal ∞-category, written informally as $(\mathcal{C}, \circ, 1)$.

Definition 2.52 (Tensored ∞-categories)**.** A *$\mathcal{C}$-tensored ∞-category* is given by a diagram of ∞-categories $\mathcal{M}^{\circledast} \xrightarrow{q} \mathcal{C}^{\circledast} \xrightarrow{p} \mathrm{N}(\Delta^{op})$ satisfying the following conditions:

a) $p \circ q : \mathcal{M}^{\circledast} \to \mathrm{N}(\Delta^{op})$ is a coCartesian fibration;
b) $q : \mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast}$ is a categorical fibration sending $(p \circ q)$-coCartesian to p-coCartesian edges;
c) For all n, the inclusion $\{n\} \subset [n]$ induces an equivalence $\mathcal{M}^{\circledast}_{[n]} \xrightarrow{\simeq} \mathcal{C}^{\circledast}_{[n]} \times \mathcal{M}^{\circledast}_{\{n\}}$.

We say that the ∞-category $\mathcal{M} \coloneqq \mathcal{M}^{\circledast}_{[0]}$ is equipped with a $\mathcal{C}$-tensored structure, written $\otimes$.

Informally, elements of $\mathcal{M}^{\circledast}_{[n]}$ correspond to tuples $(c_1, c_2, \ldots, c_n, m)$ with $c_i \in \mathcal{C}$, $m \in \mathcal{M}$; we think of the c_i's as labels of the bullets and m as a label of the $+$. The $(p \circ q)$-coCartesian lifts tensor according to the arrows; for example, the coCartesian lift of the morphism



starting at a tuple $(c_1, c_2, c_3, c_4, c_5, c_6, m)$ ends at the tuple $(1, c_2, 1, c_3 \circ c_4 \otimes c_5, c_6 \otimes m)$.

Example 2.53. Any ∞-category $\mathcal{M} = \mathcal{D}$ is naturally tensored over the monoidal ∞-category $\mathcal{C} = \mathrm{End}(\mathcal{D})$, where the tensoring evaluates functors on objects.

To formally construct this tensored structure, observe that the simplicial set $\mathcal{M} = \mathcal{D}$ is equipped with an action by the simplicial monoid $\mathcal{C} = \mathrm{End}(\mathcal{D})$.

We obtain the diagram $\mathrm{N}(\Delta^{op}) \times \Delta^1 \to \mathbf{sSet}$ drawn below.

$$\begin{tikzcd}
\cdots & \mathcal{C} \times \mathcal{C} \times \mathcal{M} \arrow[r, shift left=3] \arrow[r, shift left=1, leftarrow] \arrow[r, shift right=1] \arrow[r, shift right=3, leftarrow] \arrow[r, shift right=5] \arrow[d] & \mathcal{C} \times \mathcal{M} \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[d] & \mathcal{M} \arrow[d] \\
\cdots & \mathcal{C} \times \mathcal{C} \arrow[r, shift left=3] \arrow[r, shift left=1, leftarrow] \arrow[r, shift right=1] \arrow[r, shift right=3, leftarrow] \arrow[r, shift right=5] & \mathcal{C} \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & {[0]}
\end{tikzcd}$$

Exercise. *Applying the relative nerve construction to this diagram gives rise to an $\mathcal{C} = \mathrm{End}(\mathcal{D})$-tensored structure on $\mathcal{M} = \mathcal{C}$.*

Let $\mathcal{C}^{\circledast} \xrightarrow{p} \mathrm{N}(\Delta^{op})$ be a monoidal ∞-category and $\mathcal{M}^{\circledast} \xrightarrow{q} \mathcal{C}^{\circledast} \xrightarrow{p} \mathrm{N}(\Delta^{op})$ be a $\mathcal{C}$-tensored ∞-category. Fix an algebra object A in $\mathcal{C}$, parametrised by a section $s : \mathrm{N}(\Delta^{op}) \to \mathcal{C}^{\circledast}$ of p.

Definition 2.54 (Modules)**.** An A-module M in $\mathcal{M}$ consists of a section $s' : \mathrm{N}(\Delta^{op}) \to \mathcal{M}^{\circledast}$ with $q \circ s' = s$ and such that all morphisms drawn below are sent to $(p \circ q)$-coCartesian edges:

$$\begin{tikzcd}
- \arrow[drr] & \cdots \arrow[dr] & \bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] & \cdots & \bullet \arrow[d] & + \arrow[d] \\
 & & - & \bullet & \bullet & \bullet & \bullet & \cdots & \bullet & +
\end{tikzcd}$$

Informally, an A-module is an element $M \in \mathcal{M}$ with a multiplication map $A \otimes M \to M$ which is unital and associative up to coherent homotopy.

Definition 2.55 (Algebras over monads)**.** Given a monad T on an ∞-category $\mathcal{D}$, i.e. an algebra object in the monoidal ∞-category $\mathrm{End}(\mathcal{D})$, a T-algebra is a T-module object in the $\mathrm{End}(\mathcal{D})$-tensored ∞-category $\mathcal{D}$.

Remark 2.56. One could argue that T-algebras should be called T-modules instead, and this notational convention is indeed implemented in [Lur07]. However, we decided against this for increased consistency with the 1-categorical literature on monads.

References

[Lur07] Jacob Lurie, *Derived algebraic geometry II: Noncommutative algebra*, Preprint from the author's web page (2007).

[Lur09] ———, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659