∞-Categories and Deformation Theory

Lecture 3. The Barr–Beck–Lurie theorem and the Recognition Principle

Last lecture, we introduced monoidal ∞-categories $\mathcal{C}^{\circledast} \to \mathrm{N}(\Delta^{op})$, defined the relative nerve construction $(J \to \mathbf{sSet}) \rightsquigarrow (\mathrm{N}_F(J) \to \mathrm{N}(J))$, and used it to equip ∞-categories of endofunctors $\mathcal{C} = \mathrm{End}(\mathcal{D})$ with monoidal structures. This allowed us to define monads and their algebras in an ∞-categorical setting.

Today, we will first explain Lurie's ∞-categorical generalisation of Barr–Beck's monadicity theorem from Lecture 1, and then use it to prove the recognition principle. The ∞-categorical monadicity theorem will also be a key tool in subsequent lectures.

3.1. **A reflection on adjunctions.** We wish to construct monads from adjunctions.

For ordinary categories, this was straightforward: given an an adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ with unit η and counit ϵ, the triple $(T = GF,\ G\epsilon_F : TT \to T,\ \eta : \mathrm{id}_{\mathcal{C}} \to T)$ evidently satisfies the axioms of a monad (cf. Definition 1.35 in Lecture 1).

The corresponding construction for ∞-categories is more complicated, as we must supply an infinite amount of coherence data to specify a monad.

Adjunctions of ∞-categories can be defined in several ways:

a) Most efficiently, we can define an adjunction as a functor $F : \mathcal{C} \to \mathcal{D}$ for which the corresponding coCartesian fibration $\mathcal{M} \to \mathrm{N}(\Delta^1)$ has the property of also being Cartesian.

b) Slightly less efficiently, we could also specify both functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. However, this datum alone is overdetermined. To fix this, we must specify a unit natural transformation $u : \mathrm{id}_{\mathcal{C}} \to GF$ verifying that F and G are indeed adjoint, which means that $\mathrm{Map}_{\mathcal{D}}(FX, Y) \to \mathrm{Map}_{\mathcal{C}}(GFX, GY) \xrightarrow{u\circ -} \mathrm{Map}_{\mathcal{C}}(X, GY)$ is a weak equivalence for all X, Y.

c) Even less efficiently, we could also specify two functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ and two natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \to GF$, $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$ satisfying the natural conditions for a unit and counit. Again, this quadruple alone would be overdetermined, which we can fix by also specifying a 2-simplex $\Delta^2 \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$:

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow & \downarrow{\scriptstyle \epsilon_F} \\ & & F \end{array}$$

d) ...

Continuing in this fashion, we obtain infinitely many definitions of what an adjunction is; one can prove that all these notions are equivalent up to a contractible space of choices.

Exercise 3.1. *Given a functor $F : \mathcal{C} \to \mathcal{D}$ as in a) above, construct a functor $G : \mathcal{D} \to \mathcal{C}$ and a natural transformation $u : \mathrm{id}_{\mathcal{C}} \to GF$ satisfying the conditions specified in b).*

For most applications, the most economical definition *a)* is entirely sufficient. However, the infinitely many coherences required for a monad force us to use the "*least*" efficient definition of adjunctions, which we will explain in the following sections.

3.2. **The $(\infty,2)$-category $\{\mathcal{C},\mathcal{D}\}$.** Given an ∞-category $\mathcal{C}$, we have constructed a monoidal structure $\mathrm{End}(\mathcal{C})^{\circledast} \to \mathrm{N}(\Delta^{op})$ on $\mathrm{End}(\mathcal{C})$ in the preceeding lecture.

Shifting perspective, we may think of $\mathrm{End}(\mathcal{C})$ as a model for the full subcategory $\{\mathcal{C}\}$ of the $(\infty,2)$-category of $(\infty,1)$-categories and (not necessarily invertible) natural transformations.

To construct monads from adjunctions, we will need a similar description of the full subcategory $\{\mathcal{C},\mathcal{D}\}$ spanned by two ∞-categories $\mathcal{C},\mathcal{D}$. To this end, we proceed in three steps:

a) *Define a labelled version $\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}}$ of Δ^{op}.* We have seen that Δ^{op} is modelled by diagrams

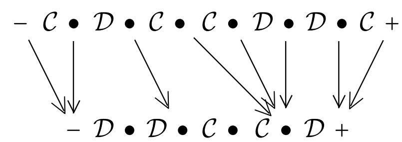
$$[n] = (\;-\quad\bullet\quad\bullet\quad\dots\quad\bullet\quad+\;),$$

with morphisms corresponding to order-preserving maps sending $-$ to $-$ and $+$ to $+$.

The objects of $\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}}$ are given by such diagrams with gaps labelled by either the symbol $\mathcal{C}$ or the symbol $\mathcal{D}$. For example, we have the following object:

$$(\;-\;\mathcal{C}\;\bullet\;\mathcal{C}\;\bullet\;\mathcal{D}\;\bullet\;\mathcal{D}\;\bullet\;\mathcal{C}\;\bullet\;\mathcal{D}\;+\;),$$

More formally, objects of $\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}}$ are given by pairs $([n]\in\Delta^{op},\; c:[n]\to\{\mathcal{C},\mathcal{D}\})$. Morphisms in $\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}}$ are order-preserving maps sending $-$ to $-$ and $+$ to $+$, which have the additional property that all gaps between two arrows carry the same label:

(7)


b) *Define a functor $F:\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}}\to\mathbf{sSet}$.* On objects, we define

$$(\;-\;\;c_0\;\bullet\;c_1\;\bullet\;\dots\;\bullet\;c_{n-1}\;\bullet\;c_n\;\;+\;)\;\;\mapsto\;\;\mathrm{Fun}(c_0,c_1)\times\dots\times\mathrm{Fun}(c_{n-1},c_n).$$

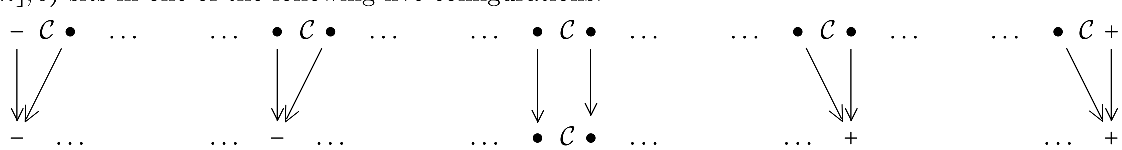
On morphisms, this functor is defined by composing functors and inserting the identities as dictated from the arrows. For example, the morphism (7) above sends an element $(\mathcal{C}\xrightarrow{F_0}\mathcal{D}\xrightarrow{F_1}\mathcal{C}\xrightarrow{F_2}\mathcal{C}\xrightarrow{F_3}\mathcal{D}\xrightarrow{F_4}\mathcal{D}\xrightarrow{F_5}\mathcal{C})$ to the element $(\mathcal{D}\xrightarrow{\mathrm{id}_{\mathcal{D}}}\mathcal{D}\xrightarrow{F_1}\mathcal{C}\xrightarrow{\mathrm{id}_{\mathcal{C}}}\mathcal{C}\xrightarrow{F_4\circ F_3\circ F_2}\mathcal{D})$.

c) *Unstraighten.* We apply the relative nerve construction introduced in Definition 4.28 of last class to obtain a coCartesian fibration $p:\mathrm{End}(\mathcal{C},\mathcal{D})^{\circledast}\to\mathrm{N}(\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}})$.

This construction will allow us access all functors between $\mathcal{C}$ and $\mathcal{D}$, and all natural transformation between such functors, in an effective way.

3.3. **Adjunction data.** We can now keep track of all higher coherence data of adjunctions. We need the following auxiliary definition:

Definition 3.2. A morphism $([n],c)\to([m],d)$ is said to be *$\mathcal{C}$-inert* if any $\mathcal{C}$-label in the domain $([n],c)$ sits in one of the following five configurations:



Definition 3.3 (Adjunction data)**.** An *adjunction datum* for a pair of ∞-categories $(\mathcal{C},\mathcal{D})$ consists of a section $s:\mathrm{N}(\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}})\to\mathrm{End}(\mathcal{C},\mathcal{D})^{\circledast}$ of $p:\mathrm{End}(\mathcal{C},\mathcal{D})^{\circledast}\to\mathrm{N}(\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}})$ sending $\mathcal{C}$-inert morphisms to p-coCartesian edges. Write $\mathrm{Adj}(\mathcal{C},\mathcal{D})$ for the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}})}(\mathrm{N}(\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}}),\mathrm{End}(\mathcal{C},\mathcal{D})^{\circledast})$ spanned by such sections.

We now unravel the information contained in an adjunction datum $\mathrm{N}(\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}}) \xrightarrow{s} \mathrm{End}(\mathcal{C},\mathcal{D})^{\circledast}$.

- First, define two functors $F := s(\ -\ \mathcal{C}\ \bullet\ \mathcal{D}\ +\ )$ and $G := s(\ -\ \mathcal{D}\ \bullet\ \mathcal{C}\ +\ )$; these will serve as left and right adjoint, respectively.
- Next, define two endofunctors $T := s(\ -\ \mathcal{C}\ \bullet\ \mathcal{C}\ +\ )$ and $\iota := s(\ -\ \mathcal{D}\ \bullet\ \mathcal{D}\ +\ )$; while T will be the monad induced by the adjunction, ι will just be a version of $\mathrm{id}_{\mathcal{D}}$.
- We obtain a natural transformation $\mathrm{id}_{\mathcal{C}} \to T$ from the following morphism:

$$\begin{array}{ccccccc} & - & & \mathcal{C} & & + & \\ \swarrow & & & & & & \searrow \\ - & \mathcal{C} & & \bullet & & \mathcal{C} & + \end{array}$$

- We obtain a natural equivalence $\mathrm{id}_{\mathcal{C}} \xrightarrow{\simeq} \iota$ from the following $\mathcal{C}$-inert morphism:

$$\begin{array}{ccccccc} & - & & \mathcal{D} & & + & \\ \swarrow & & & & & & \searrow \\ - & \mathcal{D} & & \bullet & & \mathcal{D} & + \end{array}$$

- The triangle $s(\ -\ \mathcal{C}\ \bullet\ \mathcal{D}\ \bullet\ \mathcal{C}\ +\ )$ gives two functors $F' : \mathcal{C} \to \mathcal{D}$ and $G' : \mathcal{D} \to \mathcal{C}$.

  Exercise. *Use $\mathcal{C}$-inert morphisms to produce equivalences $F' \simeq F$, $G' \simeq G$, $G'F' \simeq T$.*

- The triangle $s(\ -\ \mathcal{D}\ \bullet\ \mathcal{C}\ \bullet\ \mathcal{D}\ +\ )$ gives functors $G'' : \mathcal{D} \to \mathcal{C}$ and $F'' : \mathcal{C} \to \mathcal{D}$.

  Exercise. *Use $\mathcal{C}$-inert morphisms to produce equivalences $F'' \simeq F$, $G'' \simeq G$, and find a non-$\mathcal{C}$-inert morphism inducing a natural transformation $\epsilon : F''G'' \to \mathrm{id}_{\mathcal{D}}$; this will be the counit.*

We have produced functors F, G and natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \to GF$, $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$. A more elaborate argument, which appears as [Lur07, Lemma 3.2.9], then shows that these satisfy the axioms of an adjunction between the homotopy categories $h\mathcal{C}$ and $h\mathcal{D}$.

3.4. **The Barr–Beck–Lurie theorem.** Fix two ∞-categories $\mathcal{C}$ and $\mathcal{D}$, and consider the following maximally efficient definition of adjunctions:

Definition 3.4. A functor $F : \mathcal{C} \to \mathcal{D}$ *is a left adjoint* if the corresponding coCartesian fibration over Δ^1 is also Cartesian. Write $\mathrm{Fun}'(\mathcal{C},\mathcal{D}) \subset \mathrm{Fun}(\mathcal{C},\mathcal{D})$ for the subcategory whose objects are left adjoints and whose morphisms are natural equivalences.

It is not hard to show that any adjunction datum $s \in \mathrm{Adj}(\mathcal{C},\mathcal{D})$ determines a left adjoint $s(-\mathcal{C} \bullet \mathcal{D}+)$, and we can ask how much information is lost in this process. The following hard theorem of Lurie (cf. [Lur07, Theorem 3.2.10]) shows that the infinitely many higher coherences present in an adjunction datum can be added in an essentially unique way:

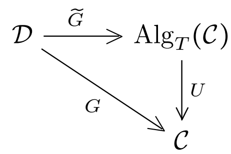
Theorem 3.5 (Adjunction data from adjunctions)**.** *Evaluation gives a trivial Kan fibration*

$$\mathrm{Adj}(\mathcal{C},\mathcal{D}) \to \mathrm{Fun}'(\mathcal{C},\mathcal{D})$$

$$s \quad \mapsto \quad s(-\mathcal{C} \bullet \mathcal{D}+)$$

Now let $F : \mathcal{C} \to \mathcal{D}$ be a left adjoint. Using Theorem 3.5 above, we pick a preimage $s \in \mathrm{Adj}(\mathcal{C}, \mathcal{D})$ from a contractible space. Write $G = s(-\mathcal{D} \bullet \mathcal{C}+)$ for the corresponding right adjoint. Restricting s to the full subcategory $\mathrm{N}(\Delta^{op}) \simeq \mathrm{N}(\Delta^{op}_{\mathcal{C}})$ of all diagrams $(-\ \mathcal{C} \bullet \ldots \bullet \mathcal{C}\ +)$ labelled only by $\mathcal{C}$ gives rise to an algebra object T in $\mathrm{End}(\mathcal{C})$.

Just like in the 1-categorical case discussed in Lecture 1, we can construct a diagram



For a formal construction of the functor $\widetilde{G}$, we refer to [Lur07, Section 3.3].

We are finally ready to state the ∞-categorical monadicity theorem:

Theorem 3.6 (Barr–Beck–Lurie, crude version)**.** *Assume that*

(1) $\mathcal{D}$ admits and G preserves geometric realisations, i.e. $\mathrm{N}(\Delta^{op})$-shaped colimits;
(2) G is conservative, i.e. if $G(f)$ is an equivalence in $\mathcal{C}$, then so is f in $\mathcal{D}$.

Then the functor $\widetilde{G} : \mathcal{D} \xrightarrow{\simeq} \mathrm{Alg}_T(\mathcal{C})$ is an equivalence of ∞-categories.

In this higher categorical result, geometric realisations play an analogous role to the reflexive coequalisers appearing in the ordinary crude Barr–Beck theorem.

To give a sharp criterion, we need a higher categorical generalisation of split coequalisers. To this end, we introduce the following enlargement of the simplex category:

Definition 3.7. The category $\Delta_{-\infty}$ has objects the finite linearly ordered sets

$$[-1] = \{\ \}\ ,\ [0] = \{0\}\ ,\ \ [1] = \{0 < 1\}\ ,\ \ [2] = \{0 < 1 < 2\}\ ,\ \ \ldots\ .$$

Morphisms $[n] \to [m]$ are given by order-preserving maps $[n] \cup \{-\infty\} \to [m] \cup \{-\infty\}$ which send $-\infty$ to $-\infty$; here $-\infty$ is defined as the least element.

Exercise 3.8.

a) Exhibit the simplex category Δ and the augmented simplex category Δ_+ as subcategories of $\Delta_{-\infty}$.
b) Show that any $\Delta_{-\infty}$-indexed diagram in an ordinary category gives a split coequaliser.

Definition 3.9 (Split simplicial objects)**.**

a) A simplicial object $X : \mathrm{N}(\Delta^{op}) \to \mathcal{C}$ in an ∞-category $\mathcal{C}$ is *split* if it extends to $\mathrm{N}(\Delta^{op}_{-\infty})$.
b) Given a functor $G : \mathcal{D} \to \mathcal{C}$, a simplicial object $X : \mathrm{N}(\Delta^{op}) \to \mathcal{D}$ is said to be *G-split* if the simplicial object $G \circ X : \mathrm{N}(\Delta^{op}) \to \mathcal{C}$ is split.

Remark 3.10. If X is a split simplicial diagram, then the restriction of X to Δ_+ is a colimit diagram; in other words, $X([-1])$ is the geometric realisation of $X|_{\mathrm{N}(\Delta^{op})}$.

Theorem 3.11 (Barr–Beck–Lurie, precise version)**.** *Given a left adjoint $F : \mathcal{C} \to \mathcal{D}$ as above, the induced functor $\mathcal{D} \to \mathrm{Alg}_T(\mathcal{C})$ is an equivalence if and only if the following conditions hold:*

(1) $\mathcal{D}$ admits and G preserves colimits of G-split simplicial diagrams in $\mathcal{D}$;
(2) G is conservative.

In practical applications, (1) is usually much harder to check than (2). In the next weeks, we will give several concrete applications of this result.

3.5. **Stable ∞-categories.** In Lecture 1, we have used the classical Barr-Beck theorem to determine when two rings R and S have equivalent categories of left modules $\mathrm{Mod}_R^\heartsuit \cong \mathrm{Mod}_S^\heartsuit$. Namely, this happens precisely if there is a compact projective generator $Q \in \mathrm{Mod}_R^\heartsuit$ with $\mathrm{End}_R(Q)^{op} \cong S$. We will now use Lurie's ∞-categorical monadicity theorem to prove a similar statement for derived ∞-categories of chain complexes.

For this application, we need to briefly discuss some further categorical constructions. The axioms for stable ∞-categories capture the key properties of derived ∞-categories of chain complexes, just like abelian categories axiomatise the key properties of ordinary categories of modules. We define:

Definition 3.12 (Stable ∞-categories)**.** An ∞-category $\mathcal{C}$ is *stable* if

a) $\mathcal{C}$ is *pointed*, which means that $\mathcal{C}$ admits an object 0 which is both initial and final;
b) Every morphism $f : X \to Y$ in $\mathcal{C}$ admits a fibre $\mathrm{fib}(f)$ and a cofibre $\mathrm{cofib}(f)$, i.e. the following pullback and pushout squares exist in $\mathcal{C}$:

$$\begin{array}{ccc} \mathrm{fib}(f) & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow f \\ 0 & \longrightarrow & Y \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & \mathrm{cofib}(f) \end{array}$$

c) A square in $\mathcal{C}$ of shape depicted below is a pullback if and only if it is a pushout.

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

These axioms are equivalent to a priori stronger conditions (cf. [Lur, Proposition 1.1.3.4]).

Proposition 3.13. *An ∞-category $\mathcal{C}$ is stable if and only if it has a zero object, admits finite limits and colimits, and a general square in $\mathcal{C}$ is a pullback if and only if it is a pushout.*

Notation 3.14. Given an object X in a pointed ∞-category $\mathcal{C}$, we will write $\Sigma X = \mathrm{cofib}(X \to 0)$ for the *suspension* of X and $\Omega X = \mathrm{fib}(0 \to X)$ for the *loop object* of X.

Exercise 3.15. *Prove that if $\mathcal{C}$ is stable, then Ω and Σ define inverse equivalences.*

We then have the following key result (cf. [Lur, Proposition 1.1.4.1]):

Proposition 3.16. *A functor $F : \mathcal{C} \to \mathcal{D}$ between stable ∞-categories preserves finite limits if and only if it preserves finite colimits.*

3.6. **Spectra.** The primeval example of a stable ∞-category is the ∞-category of spectra, which is an analogue of the category of abelian groups in ordinary category theory.

We briefly outline its construction. Write $\mathcal{S}_* = \mathcal{S}_{*/}$ for the ∞-category of pointed spaces (cf. Lecture 2, Example 2.21.c). The one-point space $*$ is a zero object in $\mathcal{S}_*$, and we obtain a loops functor $\Omega : \mathcal{S}_* \to \mathcal{S}_*$.

Definition 3.17 (Spectra)**.** The ∞-category Sp if *spectra* is given by the homotopy limit of the following tower of ∞-categories:

$$\dots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*$$

Informally, spectra are sequences of pointed spaces $X_0, X_1, \dots$ with equivalences $\Omega X_{n+1} \simeq X_n$.

We will now state several important facts about the ∞-category Sp without proof; for a comprehensive treatment of spectra in the language of ∞-categories, we refer to [Lur, Section 1.4.3].

a) The natural functor $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}_*$ admits a left adjoint $\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}$, which exhibits spectra as the *stabilisation* of spaces (the precise universal property of Sp is articulated in [Lur, Corollary 1.4.4.5]).
b) The functor Ω^∞ preserves filtered colimits, but it does *not* preserve geometric realisations.
c) Any $X \in \mathrm{Sp}$ is a canonical filtered colimit of pointed spaces $X \simeq \mathrm{colim}_n \Sigma^{\infty-n}\Omega^{\infty-n}X$, where $\Omega^{\infty-n} = \Omega^\infty \circ \Sigma^n$ and $\Sigma^{\infty-n} = \Omega^n \circ \Sigma^\infty$.
d) The ∞-categories $\mathcal{S}_*$ and Sp admit monoidal structures $\wedge$ and $\otimes$, both called *smash product*, and Σ^∞ is monoidal. In fact, both $\wedge$ and $\otimes$ define *symmetric monoidal structures*. We have not defined this notion yet, but this is a simple variation of Definition 2.41 in Lecture 2 (obtained by replacing Δ^{op} by the category of finite pointed sets Fin_*).
e) The ∞-category Sp admits a t-structure, which means that there are full subcategories $\mathrm{Sp}_{\geq 0}$ (*connective spectra*) and $\mathrm{Sp}_{\leq 0}$ (*coconnective spectra*), satisfying the following conditions:
   i) For $X \in \mathrm{Sp}_{\geq 0}$ and $Y \in \mathrm{Sp}_{\leq 0}$, we have $\mathrm{Map}_{\mathrm{Sp}}(X, \Sigma^{-1}Y) = 0$;
   ii) The functor Σ preserves $\mathrm{Sp}_{\geq 0}$ and the functor Ω preserves $\mathrm{Sp}_{\leq 0}$;
   iii) Any $X \in \mathrm{Sp}$ sits in a fibre sequence $\tau_{\geq 0}X \to X \to \tau_{\leq -1}X$ with $\tau_{\geq 0}X \in \mathrm{Sp}_{\geq 0}, \Sigma\tau_{\leq -1}X \in \mathrm{Sp}_{\leq 0}$.

   The heart $\mathrm{Sp}^\heartsuit = \mathrm{Sp}_{\geq 0} \cap \mathrm{Sp}_{\leq 0}$ of this t-structure is equivalent to N(Ab), the (nerve of the) ordinary category of abelian groups.
f) Using the monoidal structure $\otimes$ on Sp, we obtain an ∞-category Alg(Sp) of algebra objects (cf. Definition 2.49 in Lecture 2) in Sp, which are usually called $\mathbb{E}_1$*-ring spectra.*
g) The full subcategory of Alg(Sp) spanned by all objects whose underlying spectrum lies in $\mathrm{Sp}^\heartsuit$ is equivalent to the (nerve of the) ordinary category of associative rings (cf. [Lur, Proposition 7.1.3.18]). Hence, we can identify rings with discrete $\mathbb{E}_1$-ring spectra.
h) Given an $\mathbb{E}_1$-ring $A \in \mathrm{Alg(Sp)}$, Definition 2.54 from Lecture 3 gives an ∞-category Mod_A of A-module objects, which we will refer to as *A-module spectra*. Here, we have used that the monoidal ∞-category Sp is naturally tensored over itself.
i) Given objects X, Y in a general stable ∞-category $\mathcal{C}$, the space $\mathrm{Map}_{\mathcal{C}}(X,Y)$ deloops to a spectrum $\underline{\mathrm{Map}}_{\mathcal{C}}(X,Y)$, whose n^{th} space is satisfies $\Omega^{\infty-n}\underline{\mathrm{Map}}_{\mathcal{C}}(X,Y) \simeq \mathrm{Map}_{\mathcal{C}}(X, \Sigma^n Y)$. When $X = Y$, then $\underline{\mathrm{End}}_{\mathcal{C}}(X) := \underline{\mathrm{Map}}_{\mathcal{C}}(X,X)$ can be equipped with the structure of an $\mathbb{E}_1$-ring spectrum, with multiplication given by composition (cf. [Lur, Remark 7.1.2.2]).
j) If A is an ordinary ring, then Mod_A can be identified with the unbounded derived ∞-category of A, whose objects are chain complexes of A-modules $\ldots \to M_2 \to M_1 \to \ldots$. Given ordinary R-modules M, N, we have $\mathrm{Ext}_R^{-*}(M,N) \cong \pi_*\left(\underline{\mathrm{Map}}_{\mathrm{Mod}_R}(M,N)\right)$.

   We will discuss this point in more detail later.

3.7. **The** Ind**-construction.** Given an ∞-category $\mathcal{C}$, the presheaf ∞-category

$$\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$$

freely adds small colimits (cf. [Lur09, Theorem 5.1.5.6]):

Proposition 3.18 (Universal property of the presheaf category)**.** *Let $\mathcal{C}$ be a small ∞-category and $\mathcal{D}$ an ∞-category with small colimits. The Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ induces an equivalence*

$$\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

between the ∞-category of small-colimit-preserving functors $\mathcal{P}(\mathcal{C}) \to \mathcal{D}$ and the ∞-category of all functors $\mathcal{C} \to \mathcal{D}$.

A variant of the $\mathcal{P}(-)$-construction only adds *filtered* colimits (cf. Definition 2.30 in Lecture 2):

Definition 3.19 (Ind-construction)**.** Given a small ∞-category $\mathcal{C}$, let $\operatorname{Ind}(\mathcal{C}) \subset \mathcal{P}(\mathcal{C})$ be the full subcategory spanned by all functors $\mathcal{C}^{op} \to \mathcal{S}$ which preserve finite limits.

This construction satisfies the following universal property (cf. [Lur09, Proposition 5.3.5.10]

Proposition 3.20 (Universal property of the Ind-construction)**.** *Let $\mathcal{C}$ be a small ∞-category and $\mathcal{D}$ be any ∞-category containing small filtered colimits. Restriction along the Yoneda embedding $\mathcal{C}$ induces an equivalence* $\operatorname{Fun}_\omega(\operatorname{Ind}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \operatorname{Fun}(\mathcal{C}, \mathcal{D})$ *between the ∞-category* $\operatorname{Fun}_\omega(\operatorname{Ind}(\mathcal{C}), \mathcal{D})$ *of filtered-colimit-preserving functors* $\operatorname{Ind}(\mathcal{C}) \to \mathcal{D}$ *and the ∞-category of all functors $\mathcal{C} \to \mathcal{D}$.*

We now assume that $\mathcal{C}$ is an ∞-category with finite colimits, and state several key properties of the Ind-construction $\operatorname{Ind}(\mathcal{C})$:

a) The Yoneda embedding $j : \mathcal{C} \to \operatorname{Ind}(\mathcal{C})$, $X \mapsto j(X) = \operatorname{Map}_{\mathcal{C}}(X, -)$ is fully faithful, preserves finite colimits and small limits, and $j(X) \in \operatorname{Ind}(\mathcal{C})$ is compact for all $X \in \mathcal{C}$;
b) The ∞-category $\operatorname{Ind}(\mathcal{C})$ admits small colimits;
c) If $\mathcal{C}$ is stable, then so is $\operatorname{Ind}(\mathcal{C})$.
d) Any $X \in \operatorname{Ind}(\mathcal{C})$ can be obtained as a filtered colimit $X = \operatorname{colim}_a j(X_a)$ of objects $X_a \in \mathcal{C}$. If $Y = \operatorname{colim}_b j(Y_b)$ is another such object, we can compute the mapping space as

$$\begin{aligned} \operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(X, Y) &\simeq \lim_a \operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(j(X_a), Y) \\ &\simeq \lim_a \operatorname{colim}_b \operatorname{Map}_{\mathcal{C}}(j(X_a), j(Y_b)) \\ &\simeq \lim_a \operatorname{colim}_b \operatorname{Map}_{\mathcal{C}}(X_a, Y_b). \end{aligned}$$

The first equivalence is tautological, the second used that any object in the image of the Yoneda embedding is compact, and the third uses that j is fully faithful.

Definition 3.21 (Compact generation)**.** An ∞-category $\mathcal{D}$ is said to be *compactly generated* if there is a small ∞-category $\mathcal{C}$ with finite colimits and an equivalence $\mathcal{D} \simeq \operatorname{Ind}(\mathcal{C})$.

Remark 3.22. Many ∞-categories in nature are compactly generated. For example, the ∞-categories of spaces $\mathcal{S}$, spectra Sp, and module spectra Mod_R over a given $\mathbb{E}_1$-ring $R \in \operatorname{Alg}(\mathrm{Sp})$ satisfy this property. The easiest way to prove this is to exhibit all these ∞-categories as sifted-colimit-completions (cf. [Lur09, Proposition 5.5.8.10]).

Digression 3.23. *There are also various ∞-categories of interest which are not compactly generated, such as the ∞-category* $\operatorname{Shv}(\mathbb{R}, \operatorname{Mod}_k)$ *of sheaves of k-linear chain complexes on $\mathbb{R}$.*

However, most of them fit into the more general framework of presentable ∞-categories, which we shall briefly outline. Given a regular cardinal κ, we say that a simplicial set is κ-small if its set of nondegenerate simplices has cardinality less than κ. For $\kappa = \omega$, this recovers the notion of a finite simplicial set. We can then define the notion of a κ-filtered ∞-category (cf. [Lur09, Definition 5.3.1.7]*) generalising Definition 2.30 in Lecture* 2*, by allowing extensions over cones of all κ-small simplicial sets (rather than just finite ones). A generalisation of the* Ind*-construction, denoted by* $\operatorname{Ind}_\kappa$*, then freely adds κ-filtered colimits.*

An ∞-category $\mathcal{D}$ is said to be presentable if it can be written as $\mathcal{D} \simeq \operatorname{Ind}_\kappa(\mathcal{C})$ for some regular cardinal κ, where $\mathcal{C}$ is a small ∞-category containing all κ-small colimits. A list of equivalent conditions for presentability is given in [Lur09, Theorem 5.5.1.1].

Presentable ∞-categories $\mathcal{D} \simeq \operatorname{Ind}_\kappa(\mathcal{C})$ always admit small colimits; this is implied by the assumption that $\mathcal{C}$ admits κ-small colimits. If one removes this assumption, one obtains the notion of an accessible ∞-category.

3.8. **Colimit-preserving monads on Spectra.** The universal property of stabilisation (alluded to in Definition 3.17a)) implies that Sp is the free stable ∞-category generated by a single object, the *sphere spectrum* $\mathbb{S} = \Sigma^\infty(S^0)$ (cf. [Lur, Corollary 1.4.4.6.]):

Proposition 3.24. *Given a compactly generated (or in fact presentable) stable ∞-category $\mathcal{D}$, evaluation at S induces an equivalence $\mathrm{Fun}^L(\mathrm{Sp}, \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}$. Here $\mathrm{Fun}^L(\mathrm{Sp}, \mathcal{D}) \subset \mathrm{Fun}(\mathrm{Sp}, \mathcal{D})$ is the full subcategory spanned by functors which preserve small colimits.*

Taking $\mathcal{D} = \mathrm{Sp}$, we obtain an identification $\mathrm{Fun}^L(\mathrm{Sp}, \mathrm{Sp}) \xrightarrow{\simeq} \mathrm{Sp}$. The left hand side carries a natural monoidal structure given by composition, and this can be taken as a definition of the smash product $\otimes$ on the right hand side Sp. However, more work is needed to show that $\otimes$ *symmetric*; we refer to the beginning of [Lur, Section 4.8.2] for a discussion. The inverse of the above equivalence carries $X \in \mathrm{Sp}$ to $X \otimes (-)$. Passing to algebras, we deduce:

Proposition 3.25. *Evaluation at $\mathbb{S}$ induces an equivalence $\mathrm{Alg}(\mathrm{Fun}^L(\mathrm{Sp}, \mathrm{Sp})) \xrightarrow{\simeq} \mathrm{Alg}(\mathrm{Sp})$ between small-colimit-preserving monads on Sp and $\mathbb{E}_1$-ring spectra.*

If an $\mathbb{E}_1$-ring $R = T_R(\mathbb{S})$ corresponds to a monad $T_R \in \mathrm{Alg}(\mathrm{Fun}^L(\mathrm{Sp}, \mathrm{Sp}))$, then there is a canonical equivalence $\mathrm{Mod}_R \simeq \mathrm{Alg}_{T_R}(\mathrm{Sp})$.

3.9. **The Recognition Principle.** We will now develop a derived variant of Morita theory.

Let $\mathcal{C}$ be a compactly generated (or in fact presentable) stable ∞-category. Given any $Q \in \mathcal{C}$, the assignment $G_Q = \underline{\mathrm{Map}}_{\mathcal{C}}(Q, -) : \mathcal{C} \to \mathrm{Sp}$ preserves small limits. By a version of the adjoint functor theorem (cf. [Lur09, Corollary 5.5.2.9]), the functor G_Q admits a left adjoint $F_Q : \mathrm{Sp} \to \mathcal{C}$, which we will write as $F_Q(X) = X \otimes Q$. As notation suggests, the assignment $(X, Q) \mapsto F_Q(X) = X \otimes Q$ equips $\mathcal{C}$ with the structure of a Sp-tensored ∞-category. Note that by Proposition 3.24, F_Q is uniquely determined by the requirement that it preserves small colimits and sends the sphere spectrum $\mathbb{S}$ to $F_Q(\mathbb{S}) = Q$. We can now show:

Theorem 3.26 (Schwede–Shipley)**.** *Let $\mathcal{C}$ be a compactly generated (or in fact presentable) stable ∞-category . Let $Q \in \mathcal{C}$ be an object satisfying the following properties:*

a) Q is compact (cf. Lecture 2, Definition 2.33);

b) Q is a generator for $\mathcal{C}$, which means that $\underline{\mathrm{Map}}_{\mathcal{C}}(Q, D) \simeq 0$ implies $D \simeq 0$.

Then $G = \underline{\mathrm{Map}}_{\mathcal{C}}(Q, -) : \mathcal{C} \to \mathrm{Sp}$ is part of a monadic adjunction $F \dashv G$, the associated monad T preserves small colimits, and we obtain equivalences $\mathcal{C} \simeq \mathrm{Alg}_T(\mathrm{Sp}) \simeq \mathrm{Mod}_{\underline{\mathrm{End}}_{\mathcal{C}}(Q)^{op}}$.

Proof. We begin by checking that the right adjoint G (and hence T) preserves small colimits. Indeed, using Definition 3.17 c), we can write the functor G as

$$G(X) \simeq \mathrm{colim}_n \, \Sigma^{\infty-n} \Omega^{\infty-n} \underline{\mathrm{Map}}_{\mathcal{C}}(Q, X) \simeq \mathrm{colim}_n \, \Sigma^{\infty-n} \mathrm{Map}_{\mathcal{C}}(Q, \Sigma^n X).$$

Since Q is assumed to be compact, this composite of filtered-colimit-preserving functors must preserve filtered colimits. As G tautologically preserves finite (and in fact all) limits, it also preserves finite colimits Proposition 3.16. Any functor which preserves both finite and filtered colimits must preserve all small colimits.

By Proposition 3.25, the monad T is therefore given by $T(-) = R \otimes (-)$ for some $\mathbb{E}_1$-ring spectrum R. Unraveling the definition, we see that S is equivalent to $\mathrm{End}_{\mathcal{C}}(Q)^{op} = T(\mathbb{S})$, which implies the second asserted equivalence.

To prove the equivalence $\mathcal{C} \simeq \mathrm{Alg}_T(\mathrm{Sp})$, we apply Lurie's ∞-categorical Barr-Beck theorem. To verify that G is conservative, assume that G sends a morphism $f : X \to Y$ in $\mathcal{C}$ to an equivalence

$G(f) : G(X) \to G(Y)$ in Sp. Since G preserves colimits, we have $G(\mathrm{cof}(f)) \simeq \mathrm{cof}(G(f)) \cong 0$, which implies that $\mathrm{cof}(f) \simeq 0$ since Q is a generator. Hence f is an equivalence. Since G preserves small colimits, it in particular preserves geometric realisations . The (crude) Barr–Beck–Lurie theorem now shows that G induces an equivalence $\mathcal{C} \simeq \mathrm{Alg}_T(\mathrm{Sp})$. $\square$

Remark 3.27. Any equivalence $\mathrm{Mod}_R \simeq \mathrm{Mod}_S$ between module ∞-categories of $\mathbb{E}_1$-ring spectra arises as in Theorem 3.26 (cf. [Lur, Section 4.8.4]).

Remark 3.28. If R and S are ordinary rings and $Q \in \mathrm{Mod}_R$ is a compact generator of $\mathcal{C} = \mathrm{Mod}_R$ for which $\mathrm{End}_Q(R)^{op}$ is the discrete ring spectrum S, then R and S have equivalent derived ∞-categories $\mathrm{Mod}_R \simeq \mathrm{Mod}_S$. This (of course) happens whenever R and S are Morita equivalent, Schwede gives an example by considering the following two matrix rings over a field k:

$$R = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix} \middle| \, x_{ij} \in k \right\} \qquad S = \left\{ \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ 0 & y_{22} & 0 \\ 0 & 0 & y_{33} \end{pmatrix} \middle| \, y_{ij} \in k \right\}.$$

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