∞-Categories and Deformation Theory

Lecture 4. Koszul Duality for Associative Algebras and their Modules

4.1. **Koszul Duality for Modules.** Let k be a field and $A \in \mathrm{Alg}^{\mathrm{aug}}(\mathrm{Mod}_k) = \mathrm{Alg}(\mathrm{Mod}_k)_{/k}$ an augmented algebra object in Mod_k, modelled by an augmented differential graded k-algebra.

The idea behind Koszul duality is to try to use the functor

$$G(-) := \underline{\mathrm{Map}}_A(k, -) : \mathrm{Mod}_A \to \mathrm{Sp}$$

to produce an equivalence of ∞-categories, in spite of the fact that k is usually not compact (cf. Corollary 1.29 in Lecture 1). We will be able to give an explicit Koszul equivalence if A is "small" in the following sense (cf. [Lur11, Definition 3.0.1]):

Definition 4.1 (Small algebras)**.** A differential graded k-algebra $A \in \mathrm{Alg}(\mathrm{Mod}_k)$ is *small* if

a) A is connective;
b) $\dim_k(\pi_*(A)) < \infty$;
c) Writing $\mathfrak{n}$ for the radical of A, the canonical map $k \xrightarrow{\simeq} \pi_0(A)/\mathfrak{n}$ is an equivalence.

Note that small algebras are canonically augmented. The main point of interest is that the functor G above restricts to an equivalence on certain subcategories. Recall the following terminology:

Definition 4.2. A subcategory of a stable ∞-category $\mathcal{C}$ is called *thick* if it contains 0, is closed under the formation of fibres, cofibres, and retracts. Given an object $X \in \mathcal{C}$, we write $\mathrm{Thick}_{\mathcal{C}}(X)$ for the thick subcategory generated by X.

We then have (cf. [Lur11, Remark 3.4.2]):

Proposition 4.3. *The thick subcategory* $\mathrm{Thick}_A(k)$ *of* Mod_A *generated by k is equivalent to the full subcategory* $\mathrm{Coh}_A \subset \mathrm{Mod}_A$ *spanned by all* $M \in \mathrm{Mod}_A$ *for which* $\dim_k(\pi_*(M)) < \infty$.

Remark 4.4. Module spectra lying in $\mathrm{Coh}_A \subset \mathrm{Mod}_A$ are said to be *coherent*.

Proof of Proposition 4.3. Since Coh_A is evidently thick and contains k, the inclusion $\mathrm{Thick}_A(k) \subset \mathrm{Coh}_A$ is clear. For the converse, pick $M \in \mathrm{Coh}_A$ nonzero. We can then choose n minimal with $\pi_n(M) \neq 0$ and an element $x \in \pi_n(M)$ which is annihilated by the (nilpotent) augmentation ideal $\ker(\pi_0(A) \to k)$. The cofibre sequence $\Sigma^n k \xrightarrow{x} M \to M'$ induces a long exact sequence on homotopy groups, which shows that $\dim_k(\pi_*(M')) < \dim_k(\pi_*(M))$. By induction, we can therefore assume that $M' \in \mathrm{Thick}_A(k)$, which also implies the claim for M as $\Sigma^n k \in \mathrm{Thick}_A(k)$. $\qquad\square$

We deduce the following well-known result (cf. [Lur11, Remark 3.4.5]), which goes back to the work of Beilinson-Ginzburg-Soergel ([BGS96]):

Theorem 4.5. *For A a small $\mathbb{E}_1$-ring as above, there is an equivalence of stable ∞-categories* $\mathrm{Ind}(\mathrm{Coh}_A) \simeq \mathrm{Mod}_{\mathfrak{D}^{(1)}(A)^{\mathrm{op}}}$ *for* $\mathfrak{D}^{(1)}(A) := \mathrm{End}_A(k)$.

Proof. We start with the ∞-category $\mathcal{C} = \mathrm{Ind}(\mathrm{Coh}_A) \simeq \mathrm{Ind}(\mathrm{Thick}_A(k))$, which is stable as $\mathrm{Thick}_A(k)$ has this property. Like any element in the image of the Yoneda embedding, k is a compact object. To see that k is also a generator, assume that $\underline{\mathrm{Map}}_{\mathcal{C}}(k, M) = 0$ and consider the full subcategory

$$\{N \in \mathrm{Thick}_A(k) \mid \underline{\mathrm{Map}}_{\mathcal{C}}(N, M) = 0\} \subset \mathrm{Thick}_A(k).$$

Since it is thick, we deduce that any $N \in \mathrm{Thick}_A(k)$ satisfies $\underline{\mathrm{Map}}_{\mathcal{C}}(N, M) = 0$. In particular, this applies to M itself, and we have $\underline{\mathrm{Map}}_{\mathcal{C}}(M, M) = 0$. This implies that $M = 0$. $\qquad\square$

4.2. Koszul Duality for Associative Algebras and the Koszul property.

Using the Barr–Beck–Lurie theorem, we have proven that if A is a small differential graded algebra over a field k (cf. Definition 4.1), then there is an equivalence

$$\operatorname{Ind}(\operatorname{Coh}_A) \simeq \operatorname{Mod}_{\mathfrak{D}^{(1)}(A)^{\operatorname{op}}}.$$

Here $\mathfrak{D}^{(1)}(A) \simeq \mathbb{R}\operatorname{Hom}_A(k,k)$ is called the *Koszul dual* of A, satisfying $\pi_*(\mathfrak{D}^{(1)}(A)) \cong \operatorname{Ext}_A^*(k,k)$.

We will now single out a certain property of algebras known as the *Koszul property*. It is often satisfied in practice, and makes the computation of $\mathfrak{D}^{(1)}(A)$ extremely simple.

Let A be an augmented differential graded k-algebra with vanishing differentials, i.e. a (homologically) graded augmented k-algebra. Set $\overline{A} = \ker(A \to k)$.

Writing $TM = \bigoplus_{n \geq 0} M^{\otimes n}$, we consider the complex of graded A-modules

$$B(A) = \operatorname{Bar}(k, A, k) = (T(\overline{A}), d),$$

where $d([a_1|\dots|a_n]) = \sum_{i=2}^n (-1)^{\epsilon_i}[a_1|\dots|a_{i-1}a_i|\dots|a_n]$ with $\epsilon_i = (|a_1|+1) + \dots + (|a_{i-1}|+1)$.

An element $[a_1|\dots|a_n] \in (\overline{A}^{\otimes n})_i = \operatorname{Bar}_n(k,A,k)_i$ lies in "*internal degree*" $i = |a_1| + \dots + |a_n|$. Write $\operatorname{Tor}_*^A(k,k)_*$ for the bigraded A-module given by the homology of $B(A)$.

Remark 4.6. The chain complex $\mathrm{B}(A) = k \otimes_A^L k \in \operatorname{Mod}_A$ is obtained from the above chain complex of graded A-modules $B(A)$ by placing $[a_1|\dots|a_n]$ in homological degree $(|a_1|+1) + \dots + (|a_n|+1)$. Note the different fonts for B and B.

The key observation is that many algebras A as above admit an additional *Adams grading* indexed by the naturals. Write $A_i[w]$ for the component in homological degree i and Adams degree w, and assume that the augmentation induces an isomorphism $A_*[0] \cong k$.

The Bar construction then picks up a third grading satisfying

$$B(A)_n[w]_* = \bigoplus_{w_1 + \dots + w_n = w} (\overline{A}[w_1] \otimes \dots \otimes \overline{A}[w_n])_*,$$

Hence, we obtain a chain complex of bigraded A-modules.

$$\cdots$$

$$\dots \longrightarrow 0 \longrightarrow B(A)_3[3]_* \twoheadrightarrow B(A)_2[3]_* \twoheadrightarrow B(A)_1[3]_* \longrightarrow 0$$

$$\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow B(A)_2[2]_* \twoheadrightarrow B(A)_1[2]_* \longrightarrow 0$$

$$\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow B(A)_1[1]_* \longrightarrow 0$$

$$\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow k$$

Write $\operatorname{Tor}_n^A(k,k)[w]_i = \pi_n(B(A)[w]_i)$ for the component in homological degree n, internal degree i, and Adams degree w of the corresponding decomposition in homology.

Remark 4.7. More conceptually, $\operatorname{Tor}_n^A(k,k)_*[*]$ is the nth left derived functor of $k \otimes_A (-)$ on the abelian category of bigraded A-modules. This observation allows us to use other resolutions of k.

From the Bar resolution above, it is clear that $\operatorname{Tor}_n^A(k,k)_*[w]$ vanishes whenever $n > w$. The following definition of Priddy asserts that vanishing also occurs for all $n < w$:

Definition 4.8. Let A be an augmented k-algebra with a homological grading and an Adams grading as above. A is said to be *Koszul* if for all $n \neq w$, we have

$$\operatorname{Tor}_n^A(k,k)_*[w] = \ker(B(A)_n[w]_* \to B(A)_{n-1}[w]_*)/\operatorname{im}(B(A)_{n+1}[w]_* \to B(A)_n[w]_*) = 0$$

Warning 4.9. *In his original work* [Pri70], *Priddy calls these homogeneous Koszul algebras.*

For simplicity, we will assume from now on that our ground field k satisfies $\operatorname{char}(k) \neq 2$.

Definition 4.10 (Polynomial and exterior algebras)**.** If $x_1, \ldots, x_n$ are generators in Adams degree 1 and arbitrary homological degree, we define

$$k[x_1, \ldots, x_n] := T(x_1, \ldots, x_n)/(x_i \otimes x_j - x_j \otimes x_i); \qquad E[x_1, \ldots, x_n] := T(x_1, \ldots, x_n)/(x_i \otimes x_j + x_j \otimes x_i).$$

As we have *not* imposed the Koszul sign rule, $k[x_1, \ldots, x_n]$ need not be graded-commutative.

Before studying Koszul algebras in more detail, we give several simple examples.

Example 4.11. Consider $A = k[x]$ generated in Adams degree 1 and homological degree a. We use the following bigraded resolution of the A-module k:

$$\ldots \to 0 \to \Sigma^a k[x][+1] \xrightarrow{1 \mapsto x} k[x] \to 0 \to \ldots.$$

Here $[+1]$ denotes a shift by 1 in Adams grading and Σ^a is a shift by a in homological grading. Applying $k \otimes_{k[x]} (-)$, we obtain $\ldots \to 0 \to \Sigma^a k[+1] \xrightarrow{0} k \to 0 \to 0 \to \ldots$. Hence A is Koszul.

Example 4.12. Consider the exterior algebra $A = E[\epsilon] = k[\epsilon]/\epsilon^2$ on a generator in homological degree b and Adams degree 1. The bigraded A-module k admits a resolution

$$\ldots \to \Sigma^{2b}(k[\epsilon]/\epsilon^2)[+2] \xrightarrow{1 \mapsto \epsilon} \Sigma^b(k[\epsilon]/\epsilon^2)[+1] \xrightarrow{1 \mapsto \epsilon} k[\epsilon]/\epsilon^2.$$

Applying $k \otimes_A (-)$ gives $\ldots \to \Sigma^{2b} k[+2] \xrightarrow{0} \Sigma^b k[+1] \xrightarrow{0} k \to 0 \to \ldots$, hence A is Koszul.

Exercise 4.13.

a) *Show that if A, A' are Koszul algebras, then so is $A \otimes A'$.*

b) *Given generators $x_1, \ldots, x_n$ in Adams degree 1 and arbitrary homological degree, show that both $k[x_1, \ldots, x_n]$ and $E[x_1, \ldots, x_n]$ are Koszul.*

Exercise 4.14. *Prove directly that the following algebras in homological degree 0 are Koszul:*

(1) *The polynomial algebra $k[x, y]$ with x, y in Adams degree 1.*

(2) *The exterior algebra $E(x, y)$ with x, y in Adams degree 1.*

(3) *The quantum algebra $A = T(x, y)/(x \otimes y - qy \otimes x)$ for any fixed nonzero scalar $q \in k^\times$.*

4.3. **Quadratic generation.** We will prove that all Koszul algebras are of the following type:

Definition 4.15 (Quadratic algebras)**.** Given a graded k-vector space V and a homogeneous subspace $R \subset V \otimes V$, we define the following augmented k-algebra:

$$T(V; R) = T(V)/\langle R \rangle.$$

Here $\langle R \rangle$ denotes the two-sided ideal generated by the subspace R. The algebra $T(V; R)$ inherits a homological grading and an Adams grading, as the space of relations $R \subset V \otimes V$ is homogeneous for both the homological and the Adams grading on $T(V)$ (which places V in Adams degree 1).

An augmented bigraded k-algebra A is *quadratic* if $A \cong T(V; R)$ for some $V, R \subset V \otimes V$.

Proposition 4.16. *Every Koszul algebra is quadratic.*

Proof. We will prove this result in two steps (following an argument presented in [Rez12]). Given $w > 1$, the assumption $\operatorname{Tor}_1^A(k, k)[w]_* = 0$ implies that the following map is surjective:

$$B(A)_2[w]_* = \bigoplus_{\substack{w_1 + w_2 = w \\ w_i > 0}} (\overline{A}[w_1] \otimes \overline{A}[w_2])_* \longrightarrow \overline{A}[w]_* = B(A)_1[w]_*.$$

Hence by induction A is generated in Adams degree 1. Applying $\mathrm{Bar}(-)$ to the surjection $T(\overline{A}[1]) \xrightarrow{f} A$ and taking the kernel gives an exact sequence of complexes of bigraded A-modules:

$$(7) \qquad\qquad 0 \to K \to \mathrm{Bar}(T(\overline{A}[1])) \to \mathrm{Bar}(A) \to 0.$$

In degree 1 of this chain complex, our sequence is given by $0 \to K_1 \to T(\overline{A}[1]) \to A \to 0$. To prove the result, it suffices to show that the following map is surjective for all $w > 2$:

$$(8) \qquad\qquad \bigoplus_{\substack{w_1+w_2=w \\ w_i>0}} (K_1[w_1] \otimes \overline{A}[1]^{\otimes w_2} \oplus \overline{A}[1]^{\otimes w_1} \otimes K_1[w_2]) \longrightarrow K_1[w].$$

Indeed, let us restrict attention to degree 1 and degree 2 of the chain complexes in (7):

$$
\begin{array}{ccccccc}
0 \longrightarrow & K_2[w] & \longrightarrow & \bigoplus\limits_{\substack{w_1+w_2=w \\ w_i>0}} \overline{A}[1]^{\otimes w_1} \otimes \overline{A}[1]^{\otimes w_2} & \xrightarrow{f\otimes f} & \bigoplus\limits_{\substack{w_1+w_2=w \\ w_i>0}} \overline{A}[w_1] \otimes \overline{A}[w_2] \\
 & \downarrow{\scriptstyle\delta} & & \downarrow & & \downarrow \\
0 \longrightarrow & K_1[w] & \longrightarrow & \overline{A}[1]^{\otimes w} & \xrightarrow{\ f\ } & \overline{A}[w]
\end{array}
$$

Consider the natural map

$$\bigoplus_{\substack{w_1+w_2=w \\ w_i>0}} \left((K_1[w_1] \otimes \overline{A}[1]^{\otimes w_2}) \oplus (\overline{A}[1]^{\otimes j_1} \otimes K_1[w_2])\right) \xrightarrow{\ \overline{\beta}\ } \bigoplus_{\substack{w_1+w_2=w \\ w_i>0}} \overline{A}[1]^{\otimes w_1} \otimes \overline{A}[1]^{\otimes w_2}.$$

and its lift $\quad \bigoplus\limits_{\substack{w_1+w_2=w \\ w_i>0}} \left((K_1[w_1] \otimes \overline{A}[1]^{\otimes w_2}) \oplus (\overline{A}[1]^{\otimes j_1} \otimes K_1[w_2])\right) \xrightarrow{\ \beta\ } K_2[w].$

The map β is surjective. Indeed, for any decompositions $w = w_1 + w_2$, we tensor the short exact sequences $K_1[w_1] \to \overline{A}[1]^{\otimes w_1} \to \overline{A}[w_1]$ and $K_1[w_2] \to \overline{A}[1]^{\otimes w_2} \to \overline{A}[w_2]$ to obtain a diagram

$$
\begin{array}{ccccc}
K_1[w_1] \otimes K_1[w_2] & \longrightarrow & \overline{A}[1]^{\otimes w_1} \otimes K_1[w_2] & \longrightarrow & \overline{A}[w_1] \otimes K_1[w_2] \\
\downarrow & & \downarrow & & \downarrow \\
K_1[w_1] \otimes \overline{A}[1]^{\otimes w_2} & \longrightarrow & \overline{A}[1]^{\otimes w_1} \otimes \overline{A}[1]^{\otimes w_2} & \longrightarrow & \overline{A}[w_1] \otimes \overline{A}[1]^{\otimes w_2} \\
\downarrow & & \downarrow & & \downarrow \\
K_1[w_1] \otimes \overline{A}[w_2] & \longrightarrow & \overline{A}[1]^{\otimes w_1} \otimes \overline{A}[w_2] & \longrightarrow & \overline{A}[w_1] \otimes \overline{A}[w_2]
\end{array}
$$

Since k-vector spaces are flat, all columns and rows are exact. A diagram chase shows that

$$(K_1[w_1] \otimes \overline{A}[1]^{\otimes w_2}) \oplus (\overline{A}[1]^{\otimes w_1} \otimes K_1[w_2]) \longrightarrow \ker(\overline{A}[1]^{\otimes w_1} \otimes \overline{A}[1]^{\otimes w_2} \to \overline{A}[w_1] \otimes \overline{A}[w_2])$$

is surjective, which implies that the map β is surjective as well.

As $H_*(\mathrm{Bar}(k, T(\overline{A}[1]), k)[w]_*) = \mathrm{Tor}_*^{T(\overline{A}[1])}(k,k)_*[w] = 0$ for all $w > 1$, the homology long exact sequence induced by (7) shows that $H_1(K_\bullet[w]) \cong \mathrm{Tor}_2^A(k,k)_*[w] = 0$ for all $w > 2$. As $K_0[w] = 0$, this implies that $K_2[w] \xrightarrow{\delta} K_1[w]$ is surjective for all $w > 2$, and hence the following map is also surjective:

$$\bigoplus_{\substack{w_1+w_2=w \\ w_i>0}} \left((K_1[w_1] \otimes \overline{A}[1]^{\otimes w_2}) \oplus (\overline{A}[1]^{\otimes w_1} \otimes K_1[w_2])\right) \xrightarrow{\ \delta\circ\beta\ } K_1[w].$$

$\square$

4.4. **Dualising Koszul algebras.** Computing the dual of a Koszul algebra is not hard:

Theorem 4.17. *Let A be a Koszul algebra with quadratic presentation $A = T(V; R)$. Assume that $A_i[n]$ and $T(\Sigma^{-1}V^\vee; \Sigma^{-2}R^\perp)_i$ are finite-dimensional for all i, n.*

Then the Koszul dual is formal and given by $\mathfrak{D}^{(1)}(A) = \mathbb{R}\operatorname{Hom}_A(k, k) \simeq T(\Sigma^{-1}V^\vee; \Sigma^{-2}R^\perp)$, where $V^\vee = \operatorname{Map}_{\operatorname{Mod}_k}(V, k)$ and $R^\perp \subset V^\vee \otimes V^\vee$ is spanned by all $\phi \otimes \psi$ vanishing on $R \subset V \otimes V$.

Proof. Consider the differential graded coalgebra $B(A) = (T(\Sigma\overline{A}), d)$, where

$$d([a_1|\dots|a_m]) = \sum_{i=2}^n (-1)^{\epsilon_i}[a_1|\dots|a_{i-1}a_i|\dots|a_m]$$

with $\epsilon_i = (|a_1| + 1) + \dots + (|a_{i-1}| + 1)$. An element $[a_1|\dots|a_n] \in (\overline{A}^{\otimes n})_i$ lies in homological degree $i = |a_1| + \dots + |a_n| + n$ in $B(A)$. Comultiplication sends $[a_1|\dots|a_m]$ to $\sum_k [a_1|\dots|a_k] \otimes [a_{k+1}|\dots|a_m]$. The graded differential graded coalgebra $B(A) \simeq \bigoplus_w B(A)[w]$ can be dualised in two ways:

(1) Applying $\operatorname{Map}_{\operatorname{Mod}_k}(-, k)$ gives the differential graded k-algebra $\mathfrak{D}^{(1)}(A) = B(A)^\vee$;
(2) Taking the Adams-graded dual gives an Adams-graded differential graded k-algebra with

$$(\mathfrak{D}^{(1)})^{\operatorname{Gr}}(A)[n] := \operatorname{Map}_{\operatorname{Mod}_k}(B(A)[n], k).$$

These are related by a multiplicative comparison map $\bigoplus_w (\mathfrak{D}^{(1)})^{\operatorname{Gr}}(A)[w] \to \mathfrak{D}^{(1)}(A)$ given by

$$(9) \qquad \bigoplus_w B(A)[w]^\vee \longrightarrow \prod_w B(A)[w]^\vee \simeq B(A)^\vee.$$

We begin by computing $\bigoplus_w B(A)[w]^\vee$. Dualising the maps

$$\operatorname{Tor}_n^A(k, k)[n]_{*-n} \cong \ker\left(B(A)_n[n]_{*-n} \hookrightarrow B(A)_{n-1}[n]_{*-n}\right) \longrightarrow B(A)[n]_*$$

gives maps $B(A)[n]_*^\vee \to \operatorname{Ext}_A^n(k, k)[n]_{*+n}$. As A is Koszul, these assemble to an equivalence

$$\bigoplus_w B(A)[n]_*^\vee \xrightarrow{\ \simeq\ } \bigoplus_n \operatorname{Ext}_A^n(k, k)[n]_{*+n}.$$

We can represent elements in

$$\operatorname{Ext}_A^n(k, k)[n]_{*+n} \cong \operatorname{coker}\left(\bigoplus_k (\overline{A}[1]^\vee)^{\otimes k} \otimes \overline{A}[2]^\vee \otimes (\overline{A}[1]^\vee)^{\otimes(n-k-1)} \longrightarrow (\overline{A}[1]^\vee)^{\otimes n}\right)_{*+n}$$

by expressions $[\alpha_1|\dots|\alpha_n]$ with $\alpha_i \in \overline{A}[1]^\vee = V^\vee$. Here, we used $\dim_k(A_i[n]) < \infty$ for all i, n.

The product on $\bigoplus_w B(A)[w]^\vee$ corresponds to the product on $\bigoplus_w \operatorname{Ext}_A^*(k, k)[w]_{*+n}$ sending elements represented by $[\alpha_1|\dots|\alpha_k]$ and $[\alpha_{k+1}|\dots|\alpha_m]$, respectively, to $[\alpha_1|\dots|\alpha_m]$. The image of $(V^{\otimes 2}/R)^\vee \cong \overline{A}[2]^\vee \longrightarrow (\overline{A}[1]^\vee)^{\otimes 2} \cong (V^\vee)^{\otimes 2}$ is spanned by all elements $\alpha \otimes \beta$ vanishing on R. Hence $\operatorname{Ext}_A^*(k, k)[2]_{*+2} \cong ((V^\vee)^{\otimes 2}/R^\perp)_{*+2} \cong ((\Sigma^{-1}V^\vee)^{\otimes 2}/\Sigma^{-2}R_+^\perp)_*$, and more generally $\operatorname{Ext}_A^*(k, k)[w]_* \cong (\Sigma^{-1}V^\vee)^{\otimes w}/\bigcup_k \left((\Sigma^{-1}V^\vee)^{\otimes k} \otimes (\Sigma^{-2}R^\perp) \otimes (\Sigma^{-1}V^\vee)^{\otimes(w-k-1)}\right)$. These observations combine to give an equivalence

$$\bigoplus_w \operatorname{Ext}_A^*(k, k)[w]_* \simeq T(\Sigma^{-1}V^\vee; \Sigma^{-2}R^\perp).$$

Since $T(\Sigma^{-1}V^\vee; \Sigma^{-2}R^\perp)_i$ is assumed to be finite-dimensional for all i, this also shows that the comparison map (9) is an equivalence. $\qquad\qquad\square$

We illustrate Theorem 4.17 in several examples.

Example 4.18. Let V be the graded k-vector space with basis $x_1, \ldots, x_n$ in degree 0. Taking $R = \langle x_i \otimes x_j + x_j \otimes x_i \rangle$ gives the exteriour algebra $A = T(V; R) = E[x_1, \ldots, x_n]$.

Consider the dual basis $x_1^*, \ldots, x_n^*$ of $V^\vee$. An element $v^* = \sum_{i,j} \lambda_{ij} \, x_i^* \otimes x_j^* \in V^\vee \otimes V^\vee$ vanishes on R iff $\lambda_{ij} = -\lambda_{ji} = 0$ for all i, j, which happens iff $v^* \in R^\perp = \langle x_i^* \otimes x_j^* - x_j^* \otimes x_i^* \rangle$. Writing $y_i = \Sigma^{-1}(x_i^*)$, we deduce from Theorem 4.17 that

$$\mathfrak{D}^{(1)}(E[x_1, \ldots, x_n]) \simeq k[y_1, \ldots, y_n]$$

Exercise 4.19. *Prove that there is an equivalence $\mathfrak{D}^{(1)}(k[y_1, \ldots, y_n]) \simeq E[x_1, \ldots, x_n]$.*

This biduality is in fact a general phenomenon, which can be proven (under mild finiteness assumptions) using the Koszul complex. We refer to [BGS96, Secion 2.9] for a precise statement.

Remark 4.20. To see that the finiteness assumption in Theorem 4.17 is indeed necessary, consider the Koszul algebra $E[x]$ with x in Adams degree 1 and homological degree -1. Then $\mathfrak{D}^{(1)}(A) = k[[y]]$ is a power series ring generated by y in homological degree 0, while Theorem 4.17 would predict a polynomial ring.

4.5. **Poincaré Series.** Proposition 4.16 leads to the question whether every quadratic algebra is Koszul. To see that this is *not* the case, we use the following classical notion (in a form presented in [Ber14]):

Definition 4.21 (Poincaré Series)**.** Let $A = \bigoplus_{i,w} A_i[w]$ be an Adams-graded graded algebra with $\dim_k(A_i[w]) < \infty$ for all i, w. The *Poincaré series* of A is given by

$$A(t, z) = \sum_{i,w} \dim_k(A_i[w]) z^i t^w \in k[z, t].$$

We have the following criterion:

Proposition 4.22. *If $A = \bigoplus_{i,w} A_i[w]$ is a Koszul algebra as in Theorem 4.17, then*

$$P_{\mathfrak{D}^{(1)}(A)}(t, z) = P_A\left(-\frac{t}{z}, z\right)^{-1}.$$

Proposition 4.22 can be used to construct examples of quadratic algebras which are not Koszul. We work through an example of Lech:

Exercise 4.23. *Consider the following quadratic algebra with its natural Adams grading, concentrated in homological degree $i = 0$:*

$$A = k[x_1, x_2, x_3, x_4]/(x_1^2, x_2^2, x_3^2, x_4^2, x_1 x_2 + x_3 x_4)$$

(1) Show that $P_A(t, z) = 1 + 4w + 5w^2$.
(2) Compute $P_{\mathfrak{D}^{(1)}(A)}$, and use the answer to prove that A is not a Koszul algebra.

4.6. **PBW algebras.** Theorem 4.17 allows us to dualise an algebra $A = T(V; R)$ once we know that it is Koszul, but checking this property still requires some knowledge about the groups $\mathrm{Tor}_*^A(k, k)_*[*]$.

We will now introduce a simple condition on bases of V which implies that $T(V; R)$ is Koszul: Priddy's *PBW-property* (cf. [PP05, Chapter 4] for a more detailed treatment). Let V be a graded vector space with basis $x_1, \ldots x_n$, and suppose that $R \subset V \otimes V$ is a homogeneous submodule of relations. Using the lexicographic order, we define

$$S = \left\{ (i, j) \mid x_i x_j \notin \mathrm{span}(x_r x_s)_{(r,s) < (i,j)} \subset (V \otimes V)/R \right\}.$$

Exercise 4.24. *Show that the set* $\{\ x_i x_j\ |\ (i,j) \in S\ \}$ *forms a basis of* $(V \otimes V)/R$.

In particular, for any $(i,j) \notin S$, we can write

$$x_i x_j = \sum_{(r,s) < (i,j)} c_{ij}^{rs} x_r x_s \in T(V;R)$$

for uniquely determined scalars $c_{ij}^{rs} \in k$.

Definition 4.25. We say that $x_1, \ldots, x_n$ is a *PBW basis* for the quadratic algebra $T(V;R)$ if the following polynomials form a basis for $T(V;R)$:

$$\{x_{j_1} x_{j_2} \ldots x_{j_n} \in T(V;R) \ \mid \ (j_1, j_2), (j_2, j_3), \ldots, (j_{n-1}, j_n) \in S\}.$$

For $n = 0$, the above product is 1 by convention.

Priddy then established the following useful criterion:

Theorem 4.26. *If* $A = T(V;R)$ *admits a PBW-basis, then* A *is a Koszul algebra.*

Example 4.27. For $V = \langle x_1, x_2 \rangle$ and $R = \langle x_1^2, x_1 x_2 + x_2 x_1, x_2^2 \rangle$, we have $E(x_1, x_2) = T(V;R)$. We observe that $S = \{(1,2)\}$. Since $\{1, x_1, x_2, x_1 x_2\}$ is a basis for $E(x_1, x_2)$, Theorem 4.26 gives an alternative proof that $E(x_1, x_2)$ is Koszul.

Theorem 4.17 then has the following consequence:

Corollary 4.28. *Let* $x_1, \ldots, x_n$ *be a PBW-basis for* $A = T(V;R)$. *Assume that all* $A_i[n]$ *and* $T(\Sigma^{-1} V^\vee; \Sigma^{-2} R^\perp)_i$ *are finite-dimensional. Write* $x_i x_j = \sum_{(r,s)<(i,j)} c_{ij}^{rs} x_r x_s \in A$ *for* $(i,j) \notin S$. *Then* $\mathfrak{D}^{(1)}(A)$

is generated by $y_1, \ldots, y_n$ *with* y_i *in degree* $-|x_i| - 1$ *subject to the relations:*

$$(-1)^{\nu_{i,j}} y_i y_j + \sum_{(k,l) \notin S} (-1)^{\nu_{k,l}} c_{rs}^{ij} y_r y_s = 0 \quad if \quad (i,j) \in S.$$

Here $\nu_{ab} = |y_a| + (|y_a| - 1)(|y_b| - 1)$.

We conclude this lecture by stating two classical applications of Koszul algebras in topology.

4.7. **Application 1: The Homology of Loop Spaces.** Koszul duality can be used to compute the homology of loop spaces (cf. [Ber14] for a detailed treatment):

Proposition 4.29. *Let* X *be a simply connected space whose algebra of rational cochains* $C^*(X;\mathbb{Q}) \simeq H^*(X,\mathbb{Q})$ *is both formal and Koszul.*
   Then the homology of the loop space of X *is given by the Koszul dual of* $H^*(X;\mathbb{Q})$:

$$H_*(\Omega X; \mathbb{Q}) \cong \mathfrak{D}^{(1)}(H^*(X;\mathbb{Q})).$$

Example 4.30. For $X = S^2$, the cochain algebra $C^*(X, \mathbb{Q})$ is given by $E[x]$ with x in homological degree -2. As this algebra is Koszul, we deduce $H_*(\Omega S^2; \mathbb{Q}) \simeq \mathfrak{D}^{(1)}(E[x]) \simeq \mathbb{Q}[y]$ with y in degree 1. An alternative proof uses the James splitting $\Sigma \Omega \Sigma S^1 \simeq \Sigma \bigvee_{m > 0} S^m$.

4.8. **Application 2: The Adams Spectral Sequence.** We begin by recalling the Steenrod algebra, which is of key importance in topology:

Definition 4.31 (The Steenrod algebra)**.** The *Steenrod algebra* $\mathcal{A}$ (at $p = 2$) is the associative algebra generated by elements $\mathrm{Sq}^0, \mathrm{Sq}^1, \mathrm{Sq}^2, \ldots$ subject to the following relations:

(1) $\mathrm{Sq}^0 = 1$;
(2) If $i < 2j$, then $\mathrm{Sq}^i \mathrm{Sq}^j = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k} \mathrm{Sq}^k$.

Given any space X, the $\mathbb{F}_2$-valued cohomology $H^*(X, \mathbb{F}_2)$ is equipped with a natural action by the Steenrod algebra satisfying the following well-known conditions:

a) $\mathrm{Sq}^n : H^*(X, \mathbb{F}_2) \to H^{*+n}(X, \mathbb{F}_2)$ shifts degree by n;
b) $\mathrm{Sq}^n(x) = 0$ if $x \in H^m(X, \mathbb{F}_2)$ wirth $m < n$;
c) $\mathrm{Sq}^n(x) = x \cup x$ for $x \in H^n(X, \mathbb{F}_2)$;
d) $\mathrm{Sq}^n(x \cup y) = \sum_{a+b=n} \mathrm{Sq}^a(x) \cup \mathrm{Sq}^b(y)$.

Descent along the morphism of $\mathbb{E}_\infty$-rings $S \to \mathbb{F}_2$ can be used to prove the following result of Adams – we refer to [Lur10, Lecture 8] for a more detailed discussion.

Theorem 4.32 (Adams spectral sequence)**.** *There is a spectral sequence of signature*

$$E_s^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \quad \Rightarrow \quad \pi_*^s(S)_2^\wedge$$

We depict the E_2-term of this spectral sequence (in Adams convention), cf. [Rav78]:
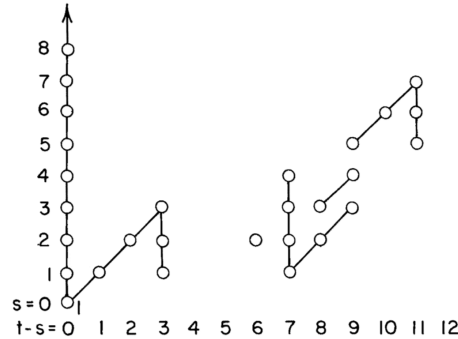
It is therefore an important computational problem to compute the algebra $Ext_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$, and our previous discussion leads to the hope that Koszul algebras might be helpful.

Since the defining relations of $\mathcal{A}$ are not homogeneous, the natural grading on the tensor algebra $T(\mathrm{Sq}^0, \mathrm{Sq}^1, \ldots)$ does *not* descend to an Adams-grading on $\mathcal{A}$. However, it induces an ascending filtration whose n^{th} stage $F^n(\mathcal{A})$ is spanned by all products of at most n generators $\mathrm{Sq}^0, \mathrm{Sq}^1, \ldots$.

The associated graded $\mathrm{Gr}(\mathcal{A})$ of this filtration admits an Adams grading, and Theorem 4.26 can be used to prove that $\mathrm{Gr}(\mathcal{A})$ is a Koszul algebra. Priddy then refines the analysis carried out in Theorem 4.17 to prove the following result:

Theorem 4.33. *The Koszul dual $\mathfrak{D}^{(1)}(\mathcal{A})$ of the Steenrod algebra is given by the Λ-algebra, which is the differential graded $\mathbb{F}_2$-algebra generated by $\lambda_1, \lambda_2, \lambda_3, \ldots$ subject to relations*

$$\lambda_a \lambda_b = \sum_{j=2b}^{\lfloor \frac{2(a+b)}{3} \rfloor} \binom{a-j-1}{j-2b} \lambda_j \lambda_{a+b-j} \quad \text{if } a \geq 2b > 0$$

with differential

$$\delta(\lambda_a) = \sum_{j=1}^{\lfloor \frac{2a}{3} \rfloor} \binom{a-j-1}{j} \lambda_j \lambda_{a-j}$$

The Λ-algebra provides a valuable tool in the computation of stable and unstable homotopy groups of spheres; we refer to [Rav03, Chapter 3] for a detailed discussion.

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