∞-Categories and Deformation Theory

Lecture 5. Koszul Duality for Commutative Algebras

Last lecture, we studied Koszul duals of associative algebras and their modules. Recall that given an augmented differential graded algebra A over a field k, we set

$$\mathfrak{D}^{(1)}(A) = \mathbb{R}\operatorname{Hom}_A(k,k), \tag{1}$$

and given a chain complex of left A-modules M, we defined

$$M \in \operatorname{Mod}_A \quad \mapsto \quad \mathbb{R}\operatorname{Hom}_A(k,M) \in \operatorname{Mod}_{\mathfrak{D}^{(1)}(A)^{\mathrm{op}}}. \tag{2}$$

There is also a *contravariant* Koszul duality functor, which is given by

$$M \in \operatorname{Mod}_A \quad \mapsto \quad \mathbb{R}\operatorname{Hom}_A(M,k) \in \operatorname{Mod}_{\mathfrak{D}^{(1)}(A)}. \tag{3}$$

However, the basic building blocks of algebraic geometry are however commutative algebras, and to study deformation theory using derived techniques, it is vital to address the following question:

Question. What is the right way of dualising an augmented *commutative* k-algebra R?

We could, of course, simply treat R as an augmented associative k-algebra and consider its Koszul dual $\mathfrak{D}^{(1)}(R)$, which is generally no longer commutative.

Exercise 5.1. *Find an example of an augmented commutative k-algebra R such that $\mathfrak{D}^{(1)}(R)$ is not commutative.*

However, the construction $R \mapsto \mathfrak{D}^{(1)}(R)$ is not optimal for the study of commutative algebras. In this lecture, we will use the commutative nature of R to define a much smaller Koszul dual $\mathfrak{D}(R)$, which will carry the structure of a (generalised) Lie algebra.

The key idea which allows us to take Koszul duals of augmented commutative algebras is that we should *not* modify the Koszul duality functor for associative algebras in (1); instead, we should generalise the contravariant Koszul duality functor for modules in (3).

At a first glance, this does not seem to make any sense – after all, the category $\mathrm{CR}_k^{\mathrm{aug}}$ of augmented commutative k-algebras is not equivalent to modules over any specific ring. Taking a closer look, however, we realise that the category $\mathrm{CR}_k^{\mathrm{aug}}$ is controlled by an augmented monad

$$\operatorname{Sym}^* = \bigoplus_n (-)^{\otimes n}_{\Sigma_n} \quad \curvearrowright \quad \operatorname{Mod}_k^{\heartsuit},$$

i.e. an augmented associative algebra object in the monoidal category of endofunctors on $\operatorname{Mod}_k^{\heartsuit}$.

As Koszul duality is an inherently derived phenomenon, we will in fact need to enlarge $\mathrm{CR}_k^{\mathrm{aug}}$ to the ∞-category $\mathrm{SCR}_k^{\mathrm{aug}} = \mathrm{CAlg}_k^{\mathrm{an,aug}}$ of *augmented simplicial commutative (or animated) k-algebras*

$$\ldots \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} R_1 \substack{\rightarrow\\ \leftarrow\\ \rightarrow} R_0.$$

To construct $\mathrm{SCR}_k^{\mathrm{aug}} = \mathrm{CAlg}_k^{\mathrm{an,aug}}$, we will need the important P_Σ-contruction, also known as *animation*, which we will introduce in Section 5.3 below.

5.1. **Warmup: Contravariant Koszul duality for modules.** But first, let us unravel the contravariant Koszul duality functor

$$M \in \mathrm{Mod}_A \quad \mapsto \quad \mathbb{R}\mathrm{Hom}_A(M,k) \in \mathrm{Mod}_{\mathfrak{D}(A)}$$

for chain complexes of modules over an augmented k-algebra A in (3) above.

This functor can be constructed in four steps:

a) Take the left derived tensor product to define a colimit-preserving functor

$$k \otimes_A^L (-) : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_k;$$

its right adjoint is restriction of scalars along the augmentation $A \to k$;

b) Postcompose with k-linear duality $(-)^\vee$ to obtain a limit-preserving functor

$$\begin{aligned} \mathrm{Mod}_A^{\mathrm{op}} &\longrightarrow \mathrm{Mod}_k \\ M \quad &\mapsto \quad (k \otimes_A^L M)^\vee \simeq \mathbb{R}\mathrm{Hom}_A(M,k); \end{aligned}$$

c) Construct a differential graded k-algebra $\mathfrak{D}^{(1)}(A) = \mathbb{R}\mathrm{Hom}_A(k,k)$;

d) Lift $\mathbb{R}\mathrm{Hom}_A(-,k) : \mathrm{Mod}_A^{\mathrm{op}} \to \mathrm{Mod}_k$ to a refined functor

$$\begin{array}{ccc} & & \mathrm{Mod}_{\mathfrak{D}^{(1)}(A)} \\ & \nearrow & \downarrow \\ \mathrm{Mod}_A^{\mathrm{op}} & \longrightarrow & \mathrm{Mod}_k. \end{array}$$

5.2. **From homological to homotopical algebra.** In step (a), we implicitly used homological algebra to replace the tensor product $k \otimes_A M$ by the more refined *derived* tensor product $k \otimes_A^L M$. Explicitly, $k \otimes_A^L M$ can be computed by first picking a projective resolution

$$\ldots \to P_2 \to P_1 \to P_0$$

of k and then setting $k \otimes_R^{\mathbb{L}} M := (\ldots \to M \otimes_R P_1 \to M \otimes_R P_0)$, thereby computing the value of the left derived functor of $k \otimes_A (-)$ on M.

To construct the desired commutative Koszul duality functor, we will need to derive the cotangent space functor defined on the category of augmented commutative k-algebras. Classical homological algebra allows us to derive additive functors $F : \mathcal{A} \to \mathcal{B}$ from an abelian category $\mathcal{A}$ with enough projectives to an abelian category $\mathcal{B}$. However:

Exercise 5.2. *Show that the category of commutative k-algebras is not abelian.*

To overcome this difficulty, we make use Quillen's *homotopical algebra*, cf. [Qui06]. The basic idea is straightforward: in homological algebra, we replaced modules by connective chain complexes (equivalently simplicial modules) to left derive functors defined on modules; in homotopical algebra, we replace commutative algebras by *simplicial commutative (= animated) algebras* in order to derive functors defined on commutative algebras.

5.3. **Animated rings.** Fix a commutative ring A for the remainder of this lecture. The ∞-category $\mathrm{SCR}_A = \mathrm{CAlg}_A^{\mathrm{an}}$ of simplicial commutative (= animated) A-algebras can be defined in two ways:

(1) we can equip the category of simplicial objects in CR_A, the category of commutative A-algebras, with a suitable model structure and then pass to the underlying ∞-category;

(2) we can freely adjoin filtered colimits and geometric realisations to the category Poly_A of polynomial A-algebras $A[X_1, \ldots, X_n]$.

Today, we will follow the second approach, as it will serve as an excuse to learn about an important ∞-categorical technique known as *animation*. We have already encountered two ways of formally adjoining certain colimits:

(1) If $\mathcal{C}$ is a small ∞-category, the presheaf ∞-category $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ freely adds small colimits.
(2) If $\mathcal{C}$ admits finite limits, the ∞-category $\mathrm{Ind}(\mathcal{C}) = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ of finite-limit-preserving presheaves freely adds filtered colimits to $\mathcal{C}$.

We will now introduce a third construction in this vein, which simultaneously adjoins filtered colimits and geometric realisations:

Definition 5.3 (Animation)**.** Given a small ∞-category $\mathcal{C}$ with finite coproducts, we let

$$\mathcal{P}_\Sigma(\mathcal{C}) \subset \mathcal{P}(\mathcal{C})$$

be the full subcategory spanned by all functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ which preserve finite products.

Recall from Definition 2.33 in Lecture 2 that an object is called *compact* if mapping out of it preserves filtered colimits. There is an analogous notion for geometric realisations:

Definition 5.4 (Projective object)**.** Let $\mathcal{C}$ be an ∞-category with geometric realisations. An object $X \in \mathcal{C}$ is called *projective* if the functor $\mathrm{Map}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathcal{S}$ preserves geometric realisations.

Animation has the following universal property (cf. [Lur09, Theorem 5.5.8.15]):

Proposition 5.5 (Universal property of animation)**.** *Let $\mathcal{C}$ be a small ∞-category and let $\mathcal{D}$ be any ∞-category containing filtered colimits and geometric realisations.*

Restriction along the Yoneda embedding induces an equivalence

$$\mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

between $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ *and the full subcategory* $\mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D}) \subset \mathrm{Fun}(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D})$ *spanned by those functors $\mathcal{P}_\Sigma(\mathcal{C}) \longrightarrow \mathcal{D}$ which preserve filtered colimits and geometric realisations.*

Notation 5.6 (Nonabelian left derived functors)**.** Given a functor $f : \mathcal{C} \longrightarrow \mathcal{D}$ in the above situation, we call the corresponding functor $Lf : \mathcal{P}_\Sigma(\mathcal{C}) \longrightarrow \mathcal{D}$ the nonabelian left derived functor of f.

Remark 5.7. By [Lur09, Corollary 5.5.8.17], a functor F preserves geometric realisations and filtered colimits if and only if it preserves *sifted colimits*. These are colimits of diagrams $D : K \to \mathcal{C}$ where the simplicial set K is nonempty and the diagonal $K \to K \times K$ is cofinal, cf. [Lur09, 5.5.8.1].

The universal property in Proposition 5.5 is extremely helpful in applications, as it allows us to easily construct functors out of categories of the form $\mathcal{P}_\Sigma(\mathcal{C})$. Two examples will be particularly important in the following lectures: the cotangent complex and the derived de Rham complex.

It is therefore important to identify when a given ∞-category takes this special form. We state the following criterion (cf. [Lur09, Proposition 5.5.8.22]):

Proposition 5.8. *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor from a small ∞-category $\mathcal{C}$ with finite coproducts to an ∞-category $\mathcal{D}$ with filtered colimits and geometric realisations.*

Then $Lf : \mathcal{P}_\Sigma(\mathcal{C}) \longrightarrow \mathcal{D}$ is an equivalence if and only if the following conditions hold:

(1) the functor f is fully faithful;
(2) the essential image of f consists of compact projective objects of $\mathcal{D}$
(3) the essential image of f generates $\mathcal{D}$ under filtered colimits and geometric realisations.

Remark 5.9. If only conditions (1) and (2) hold, then Lf is automatically fully faithful.

Exercise 5.10. *Write* $\mathrm{Vect}_A^{\mathrm{ff}}$ *for the ordinary category of finite free* A*-modules. Construct an equivalence between* $\mathcal{P}_\Sigma(\mathrm{Vect}_A^\omega)$ *and the full subcategory* $\mathrm{Mod}_{A,\geq 0} \subset \mathrm{Mod}_A$ *spanned by all connective chain complexes over* A*.*

We use animation to define simplicial commutative algebras:

Definition 5.11. Let $\mathrm{Poly}_A \subset \mathrm{CR}_A$ be the full subcategory spanned by all commutative A-algebras of the form

$$A[x_1, \dots, x_n].$$

The ∞-category of *simplicial commutative* A*-algebras* (or *animated* A-algebras) is defined as

$$\mathrm{CAlg}_A^{\mathrm{an}} = \mathrm{SCR}_A \coloneqq \mathcal{P}_\Sigma(\mathrm{Poly}_A)$$

Exercise 5.12.

a) Define a model structure on the category $\mathbf{SCR}_A$ *of simplicial objects in commutative* A*-algebras whose weak equivalences are weak equivalences of underlying simplicial sets and whose fibrations are levelwise surjections.*
b) Prove that the underlying ∞*-category of* $\mathbf{SCR}_A$ *is equivalent to* $\mathrm{CAlg}_A^{\mathrm{an}} = \mathcal{P}_\Sigma(\mathrm{Poly}_A)$*.*

Exercise 5.13. *For* k *a field, let* $\mathbf{cdga}_k$ *be the category of commutative differential graded* k*-algebras.*

(1) Show that if $\mathrm{char}(k) = 0$*, the category* $\mathbf{cdga}_k$ *admits a model structure whose weak equivalences are given by quasi-isomorphisms and whose fibrations are given by levelwise surjections. Writing* cdga_k *for the underlying* ∞*-category of* $\mathbf{cdga}_k$*, show that* $\mathrm{CAlg}_k^{\mathrm{an}}$ *is equivalent to full subcategory* $\mathrm{cdga}_{k,\geq 0} \subset \mathrm{cdga}_k$ *spanned by all connective objects.*
(2) Show that if $\mathrm{char}(k) = p$*, the category* $\mathbf{cdga}_k$ *does not admit a model structure whose weak equivalences are given by quasi-isomorphisms and whose fibrations are levelwise surjections.*

Let $A \in \mathrm{CR}$ again be a general commutative ring. We need two variants of Definition 5.11:

Variant 5.14. *Let* $\mathrm{Poly}_A^{\mathrm{aug}}$ *be the category of augmented commutative* A*-algebras of the form*

$$A[x_1, \dots, x_n] \to A$$

with morphisms given by those maps of A*-algebras which commute with the augmentation. The* ∞*-category of augmented simplicial commutative* A*-algebras (or animated nonunital* A*-algebras) is then defined as* $\mathrm{CAlg}_A^{\mathrm{an,aug}} \coloneqq \mathcal{P}_\Sigma(\mathrm{Poly}_A^{\mathrm{aug}})$*.*

Variant 5.15. *Let* $\mathrm{Poly}_A^{\mathrm{nu}}$ *be the category of nonunital commutative* A*-algebras of the form*

$$\mathrm{I}\, A[x_1, \dots, x_n] = \ker(A[x_1, \dots, x_n] \to A).$$

The ∞*-category of nonunital simplicial commutative* A*-algebras (or animated nonunital* A*-algebras) is defined as* $\mathrm{CAlg}_A^{\mathrm{an,nu}} \coloneqq \mathcal{P}_\Sigma(\mathrm{Poly}_A^{\mathrm{nu}})$*.*

Exercise 5.16.

(1) Show that the augmentation ideal functor $\mathrm{I} : \mathrm{CAlg}_A^{\mathrm{an,aug}} \to \mathrm{CAlg}_A^{\mathrm{an,nu}}$ *is an equivalence.*
(2) Define the forgetful functor $\mathrm{forget}^{\mathrm{nu}} : \mathrm{CAlg}_A^{\mathrm{an,nu}} \to \mathrm{Mod}_{A,\geq 0}$ *and show that it is part of a monadic adjunction* $\mathrm{free}^{\mathrm{nu}} \dashv \mathrm{forget}^{\mathrm{nu}}$*. Write* $\mathbb{L}\,\mathrm{Sym}_A^{\mathrm{nu}}$ *for the corresponding monad.*
(3) Show that the underlying functor of $\mathbb{L}\,\mathrm{Sym}_A^{\mathrm{nu}}$ *is given by* $\bigoplus_{n>0} \mathbb{L}\,\mathrm{Sym}_A^n$*, where* $\mathbb{L}\,\mathrm{Sym}_A^n$ *is the left derived functor of the* n^{th} *symmetric power functor* $M \mapsto M_{\Sigma_n}^{\otimes n}$*.*

5.4. **The cotangent fibre.** The formalism of nonabelian left derived functors will allow us to construct the desired Koszul duality functor for commutative rings.

To begin with, recall that in Step a) of Section 5.1, the augmentation $A \to k$ gave rise to the restriction-of-scalars functor $\mathrm{Mod}_k \to \mathrm{Mod}_A$, whose left adjoint $k \otimes_A^L (-)$ was the main ingredient for contravariant Koszul duality for modules.

In the commutative setting, we note that, given some $A \in \mathrm{CR}$, the monad

$$\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}} = \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}_A = \bigoplus_{n>0} \mathbb{L}\,\mathrm{Sym}^n_A$$

parametrising nonunital simplicial commutative A-algebras is augmented over the identity monad $\mathbf{1} = \mathbb{L}\,\mathrm{Sym}^1_A$, so that restriction along $\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}_A \to \mathbf{1}$ defines a functor $\mathrm{sqz}^{\mathrm{nu}} : \mathrm{Mod}_{A,\geq 0} \to \mathrm{CAlg}^{\mathrm{an,nu}}_A$.

Exercise 5.17.

a) Use the equivalence $\mathrm{Mod}_{A,\geq 0} \simeq \mathcal{P}_\Sigma(\mathrm{Vect}^{\mathrm{ff}}_A)$ *to construct the (nonunital) trivial square-zero algebra functor* $\mathrm{sqz}^{\mathrm{nu}}$ *rigorously.*

b) Show that $\mathrm{sqz}^{\mathrm{nu}}$ *admits a left adjoint* $\mathrm{cot}^{\mathrm{nu}}$ *and construct an equivalence* $\mathrm{cot}^{\mathrm{nu}} \circ \mathrm{free}^{\mathrm{nu}} \simeq \mathrm{id}$*, where* $\mathrm{free}^{\mathrm{nu}} : \mathrm{Mod}_{A,\geq 0} \to \mathrm{CAlg}^{\mathrm{an,nu}}_A$ *is left adjoint to the forgetful functor* $\mathrm{forget}^{\mathrm{nu}}$.

Definition 5.18 (Cotangent fibre)**.** The *cotangent fibre* of an augmented simplicial commutative (= animated) A-algebra $R \in \mathrm{CAlg}^{\mathrm{an,aug}}_A$ is given by

$$\mathrm{cot}(R) \coloneqq \mathrm{cot}^{\mathrm{nu}}(\mathrm{I}\,R),$$

where I is the augmentation ideal functor from Exercise 5.16.

Remark 5.19. Given $R \in \mathrm{CAlg}^{\mathrm{an,aug}}_A$ and $M \in \mathrm{Mod}_{A,\geq 0}$, we have an equivalence

$$\mathrm{Map}_{\mathrm{Mod}_{A,\geq 0}}(\mathrm{cot}(R), M) \simeq \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{an,aug}}_A}(R, \mathrm{sqz}(M)),$$

where $\mathrm{sqz}(M) = A \oplus \mathrm{sqz}^{\mathrm{nu}}(M)$ is the unital square-zero extension of A by M.

The cohomotopy groups $\mathrm{AQ}^*(R) \coloneqq \pi^*(\mathrm{cot}(R))$ are known as the *André–Quillen cohomology groups* of the ring R. We see from the above adjunction that

$$\mathrm{AQ}^n(R) = \mathrm{Map}_{\mathrm{Mod}_{A,\geq 0}}(\mathrm{cot}(R), A[n]) \simeq \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{an,aug}}_A}(R, \mathrm{sqz}(A[n])).$$

In other words, $\mathrm{AQ}^*(R)$ measures how R maps into trivial square-zero extensions.

In these lectures, will review several methods for computing cotangent fibres. The most general such technique uses the so-called *bar construction.* Indeed, given an augmented simplicial commutative A-algebra $R \in \mathrm{CAlg}^{\mathrm{an,aug}}$ with augmentation ideal $\mathrm{I}\,R \in \mathrm{CAlg}^{\mathrm{an,nu}}$, we consider the following simplicial object in $\mathrm{CAlg}^{\mathrm{an,nu}}$:

$$\mathrm{Bar}_\bullet(\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}, \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}, \mathrm{I}\,R) = \Big(\dots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}(\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}(\mathrm{I}\,R)) \;\substack{\longrightarrow\\\longrightarrow}\; \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}(\mathrm{I}\,R) \Big)$$

Exercise 5.20. *Use the structure map* $\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}(\mathrm{I}\,R) \to \mathrm{I}\,R$ *of* $\mathrm{I}\,R$*, the monadic multiplication map* $\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}} \circ \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}} \to \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}$*, and then unit* $\mathrm{id} \to \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}$ *to define all morphisms in this diagram.*

Using the structure map of $\mathrm{I}\,R$, we can extend $\mathrm{Bar}_\bullet(\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}, \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}, \mathrm{I}\,R)$ to an *augmented* simplicial object in the ∞-category $\mathrm{CAlg}^{\mathrm{an,nu}}_A$.

Proposition 5.21. *The following induced map is an equivalence:*

$$|\mathrm{Bar}_\bullet(\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}, \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}, \mathrm{I}\,R)| \coloneqq \mathrm{colim}_{\Delta^{\mathrm{op}}} \mathrm{Bar}_\bullet(\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}, \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}, \mathrm{I}\,R) \xrightarrow{\simeq} \mathrm{I}\,R.$$

Proof. By Exercise 5.16 (3), $\mathbb{L}\mathrm{Sym}^{\mathrm{nu}}$ preserves geometric realisations. By [Lur07, Corollary 2.3.7], this implies that $\mathrm{forget}^{\mathrm{nu}}$ detects geometric realisations, so it suffices to show that the diagram $|\mathrm{Bar}_\bullet(\mathbb{L}\mathrm{Sym}^{\mathrm{nu}}, \mathbb{L}\mathrm{Sym}^{\mathrm{nu}}, \mathrm{I}\,R)| \to \mathrm{I}\,R$ is a colimit diagram in Mod_A. But in Mod_A, this augmented simplicial diagram admits an extra degeneracy, which implies that it is a colimit diagram. □

Corollary 5.22. *Given $R \in \mathrm{CAlg}_A^{\mathrm{an,aug}}$, there is an equivalence*

$$\cot(R) \simeq |\mathrm{Bar}_\bullet(\mathrm{id}, \mathbb{L}\mathrm{Sym}^{\mathrm{nu}}, \mathrm{I}\,R)|$$

Proof. As cot is a left adjoint and therefore preserves colimits, this follows immediately from Exercise 5.17 and Proposition 5.21. □

5.5. **From associative to commutative Koszul duality.** To construct Koszul duality in the commutative setting, we make the following substitutes in Section 5.1:

Associative algebra k	$\rightsquigarrow$	Identity monad $\mathbf{1} = \mathbb{L}\mathrm{Sym}_k^1$ on $\mathrm{Mod}_{k,\geq 0}$
Augmented algebra A	$\rightsquigarrow$	Augmented monad $\mathbb{L}\mathrm{Sym}_k^{\mathrm{nu}}$ on $\mathrm{Mod}_{k,\geq 0}$
∞-category Mod_A of chain complexes over A	$\rightsquigarrow$	∞-category $\mathrm{CAlg}_k^{\mathrm{an,nu}}$ of nonunital simplicial commutative k-algebras
Restriction of scalars functor $\mathrm{Mod}_k \to \mathrm{Mod}_A$ defined via $A \to k$	$\rightsquigarrow$	Trivial algebra functor $\mathrm{sqz}^{\mathrm{nu}} : \mathrm{Mod}_{k,\geq 0} \to \mathrm{CAlg}_k^{\mathrm{an,nu}}$ defined via $\mathbb{L}\mathrm{Sym}_k^{\mathrm{nu}} \to \mathbf{1}$
Extension of scalars functor $k \otimes_A^L (-) : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_k$	$\rightsquigarrow$	Cotangent fibre functor $\cot^{\mathrm{nu}} : \mathrm{CAlg}_k^{\mathrm{an,nu}} \to \mathrm{Mod}_{k,\geq 0}$

Using the constructions introduced earlier, we can generalise step (a) and (b) in Section 5.1 to the setting of nonunital animated k-algebras as follows:

(a') Take the colimit-preserving cotangent fibre functor

$$\cot^{\mathrm{nu}} : \mathrm{CAlg}_k^{\mathrm{an,nu}} \longrightarrow \mathrm{Mod}_{k,\geq 0}.$$

Its right adjoint is the trivial algebra functor $\mathrm{sqz}^{\mathrm{nu}} : \mathrm{Mod}_{k,\geq 0} \to \mathrm{CAlg}_k^{\mathrm{an,nu}}$;

(b') Postcompose with k-linear duality $(-)^\vee$ to obtain a limit-preserving functor

$$(\mathrm{CAlg}_k^{\mathrm{an,nu}})^{\mathrm{op}} \longrightarrow \mathrm{Mod}_k\ , \qquad R \mapsto \cot(R)^\vee$$

The rest of this class is dedicated to also generalising steps (c) and (d) to the commutative setting.

5.6. **The naive Koszul dual monad.** In step (c) of Section 5.1, we defined the Koszul dual of an augmented associative algebra A as $\mathfrak{D}^{(1)}(A) = \mathbb{R}\operatorname{Hom}_A(k,k)$. Our next goal is to construct a well-behaved Koszul dual monad $\mathfrak{D}(\mathbb{L}\operatorname{Sym}_k^{\mathrm{nu}})$ of the augmented monad $\mathbb{L}\operatorname{Sym}_k^{\mathrm{nu}}$.

To this end, we begin by observing that the functor

$$(\mathrm{CAlg}_k^{\mathrm{an,nu}})^{\mathrm{op}} \longrightarrow \mathrm{Mod}_{k,\leq 0}\ , \qquad R \mapsto \cot^{\mathrm{nu}}(R)^\vee$$

preserves limits; its right adjoint is given by the assignment $V \mapsto \mathrm{sqz}^{\mathrm{nu}}(V^\vee)$.

By abstract nonsense, this adjunction gives rise to a canonical monad

$$T^{\mathrm{naive}}(-) = (\cot^{\mathrm{nu}}(\mathrm{sqz}^{\mathrm{nu}}((-)^\vee)))^\vee$$

on the full subcategory $\mathrm{Mod}_{k,\leq 0} \subset \mathrm{Mod}_k$ of coconnective chain complexes. It suffers from two defects:

i) It is only defined on coconnective complexes, so will not recover differential graded Lie algebras;
ii) It does not preserve sifted colimits, which, as we will see later, is a problem for applications in deformation theory.

To circumvent these obstacles, we will replace T^{naive} by a more well-behaved monad Lie_k^π, which is obtained by left Kan extending a certain restriction of T^{naive}. Making this idea precise will be the goal of the rest of this lecture, and will require several preliminaries.

5.7. **Cotangent fibres via partition complexes.** We start by giving a concrete expression for cotangent fibres of trivial square-zero extensions in terms of the following simplicial sets:

Definition 5.23 (Doubly suspended partition complexes)**.** For each $n \geq 0$, we define a simplicial Σ_n-set $P(n)$ by specifying its set of k-simplices as

$$P(n)_k = \left\{ [\hat{0} = \sigma_0 \leq \sigma_1 \leq \ldots \leq \sigma_k = \hat{1}] \ \middle|\ \sigma_i \text{ are partitions of } \{1,\ldots,n\} \right\} \ \coprod\ \{*\},$$

where $\hat{0}$ is the discrete partition and $\hat{1}$ is the indiscrete partition of the set $\{1,\ldots,n\}$.

Degeneracy maps insert repeated partitions into chains and fix $*$. Face maps delete partitions from chains whenever this yields a "legal" chain starting in $\hat{0}$ and ending in $\hat{1}$, and otherwise map to $*$.

Let us fix a simplicial k-vector space $V_\bullet$ with associated chain complex $|V_\bullet| \in \mathrm{Mod}_{k,\geq 0}$. As cot preserves geometric realisations, we obtain, by Corollary 5.22, the following equivalence:

$$\begin{aligned}\cot^{\mathrm{nu}}(\mathrm{sqz}^{\mathrm{nu}}(|V_\bullet|)) &\simeq |\mathrm{Bar}_\bullet(\mathrm{id}, \mathbb{L}\operatorname{Sym}^{\mathrm{nu}}, |V_\bullet|)| \\ &\simeq \left\| \ \cdots \ \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} \ \bigoplus_{m\geq 1}\Big(\bigoplus_{n\geq 1}(V_\bullet)^{\otimes n}_{\Sigma_n}\Big)^{\otimes m}_{\Sigma_m} \ \substack{\rightarrow\\ \leftarrow\\ \rightarrow} \ \bigoplus_{n\geq 1}(V_\bullet)^{\otimes n}_{\Sigma_n} \ \substack{\rightarrow\\ \leftarrow} \ V_\bullet \ \right\| .\end{aligned}$$

For $(X,*)$ a pointed set, let $k[X]$ be the free k-module on X subject to the relation $0 \simeq *$.

We will now relate the simplicial sets $P(n)_\bullet$, defined using partitions, to the cotangent fibre of trivial square-zero extensions:

Exercise 5.24. *Let $V_\bullet$ be a simplicial k-vector space with associated chain complex $V = |V_\bullet|$.*

(1) Expand symmetric powers binomially to prove that $\cot^{\mathrm{nu}}(\mathrm{sqz}^{\mathrm{nu}}(V))$ is equivalent to the realisation of the following bisimplicial set:

$$\left\| \ \cdots \ \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} \ \bigoplus_{n\geq 1} k[P(n)_2] \underset{\Sigma_n}{\otimes} (V_\bullet)^{\otimes n} \ \substack{\rightarrow\\ \leftarrow\\ \rightarrow} \ \bigoplus_{n\geq 1} k[P(n)_1] \underset{\Sigma_n}{\otimes} (V_\bullet)^{\otimes n} \ \substack{\rightarrow\\ \leftarrow} \ \bigoplus_{n\geq 1} k[P(n)_0] \underset{\Sigma_n}{\otimes} (V_\bullet)^{\otimes n} \ \right\|,$$

(2) Deduce the following equivalence:

$$\cot^{\mathrm{nu}}(\mathrm{sqz}^{\mathrm{nu}}(V)) \simeq \bigoplus_{n\geq 1} \widetilde{C}_\bullet(|P(n)|,k) \underset{\Sigma_n}{\otimes} (V_\bullet)^{\otimes n}.$$

Here $\widetilde{C}_\bullet(|P(n)|,k)$ *are the* k*-valued chains on the geometric realisation* $|P(n)|$ *of* $P(n)$*.*

Using the above formula and the topology of partition complexes, one can show:

Proposition 5.25. *If* V *belongs to* $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0} \subset \mathrm{Mod}_k$*, the full subcategory spanned by all connective chain complexes with* $\dim(\pi_i(V)) < \infty$*, then* $\cot^{\mathrm{nu}}(\mathrm{sqz}^{\mathrm{nu}}(V)) \in \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ *shares the same property.*

5.8. **The partition Lie algebra monad.** We return to our goal of replacing the naive Koszul dual monad T^{naive} of $\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}_k$ on $\mathrm{Mod}_{k,\leq 0}$ by a more well-behaved monad Lie^π_k defined on all of Mod_k. We will only sketch its construction:

Construction 5.26 (Partition Lie algebra monad)**.**

1) Proposition 5.25 implies that the monad T^{naive} preserves the full subcategory $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0} \subset \mathrm{Mod}_k$. Hence the restriction $T^{\mathrm{naive}}|_{\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}}$ acquires the structure of a monad. Exercise 5.24(2) gives a quite explicit description: if $V^\bullet$ is a cosimplicial k-module whose associated chain complex $\mathrm{Tot}(V^\bullet)$ is of finite type, then

$$T|_{\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}}(\mathrm{Tot}(V^\bullet)) \simeq \bigoplus_n \mathrm{Tot}\left(\widetilde{C}^\bullet(|P(n)_\bullet|,k) \otimes (V^\bullet)^{\otimes n}\right)^{\Sigma_n}.$$

Here $\widetilde{C}^\bullet(|P(n)_\bullet|,k)$ are the k-valued cosimplices of $|P(n)_\bullet|$, the functor $(-)^{\Sigma_n}$ takes strict fixed points, and the tensor product is computed in cosimplicial k-modules.

2) We then check that the functor $T|_{\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}}$
   - *is right complete*, which means that the canonical map $\mathrm{colim}_n\, T(\tau_{\leq -n}V) \xrightarrow{\simeq} T(V)$ is an equivalence for all $V \in \mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$;
   - *preserves finite coconnective geometric realisations*, which means that if $V_\bullet$ is a simplicial object in $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ which is m-skeletal (for some m) and with $|V_\bullet| \in \mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$, then the canonical map $|T(V_\bullet)| \xrightarrow{\simeq} T(|V_\bullet|)$ is an equivalence.

3) Let $\mathrm{End}_\Sigma^{\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}} \subset \mathrm{End}(\mathrm{Mod}_k)$ be the full subcategory of sifted-colimit-preserving endofuctors of Mod_k which preserve $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$. Let $\mathrm{End}'_\sigma(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}) \subset \mathrm{End}(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0})$ be the full subcategory of right complete endofunctors of $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ preserving finite coconnective geometric realisations. By [BM19, Corollary 3.17], the following monoidal restriction functor is an equivalence:

$$\mathrm{End}_\Sigma^{\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}}(\mathrm{Mod}_k) \xrightarrow{\simeq} \mathrm{End}'_\sigma(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}).$$

4) Using this equivalence, we extend the monad $T|_{\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}}$ from part (a) to obtain the monad Lie^π_k on Mod_k. This monad Lie^π_k preserves filtered colimits and geometric realisations, and if $V^\bullet$ is a cosimplicial k-module with associated chain complex $\mathrm{Tot}(V^\bullet)$, then

$$\mathrm{Lie}^\pi_k(\mathrm{Tot}(V^\bullet)) \simeq \bigoplus_n \mathrm{Tot}\left(\widetilde{C}^\bullet(\Sigma|\Pi_n|^\diamond,K) \otimes (V^\bullet)^{\otimes n}\right)^{\Sigma_n}.$$

Definition 5.27. The ∞*-category of partition Lie algebras* is the ∞-category of algebras over the monad Lie^π_k. We will denote this ∞-category by $\mathrm{Alg}_{\mathrm{Lie}^\pi_k}$.

We have generalised step (c) in Section 5.1:

(c') Construct the monad $\mathrm{Lie}^\pi_k = \mathfrak{D}(\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}_k)$ from the augmented monad $\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}_k$.

To generalise step (d), we want to promote

$$\cot(A)^\vee$$

to a partition Lie algebra for any augmented animated k-algebra A.

To this end, we use the category $\mathrm{Poly}^{\mathrm{aug}}_k$ of augmented commutative k-algebras of the form $k[x_1, \dots, x_n] \to k$, see Variant 5.14. By [Lur04, Proposition 3.2.14], we know that $\cot(A)^\vee$ belongs to $\mathrm{Mod}^{\mathrm{ft}}_{k,\le 0}$ for any $A \in \mathrm{Poly}^{\mathrm{aug}}_k$. Hence the tautological T^{naive}-algebra structure on

$$\cot^{\mathrm{aug}}(A)^\vee \simeq \cot^{\mathrm{nu}}(\mathrm{I}A)^\vee$$

equips this chain complex with a Lie^π_k-algebra structure. We obtain a functor

$$\mathrm{Poly}^{\mathrm{aug}}_k \to \mathrm{Alg}^{\mathrm{op}}_{\mathrm{Lie}^\pi_k}$$

and define:

(d') The Koszul duality functor

$$\mathfrak{D} : \mathrm{SCR}^{\mathrm{aug}}_k \simeq \mathcal{P}_\Sigma(\mathrm{Poly}^{\mathrm{aug}}_k) \longrightarrow \mathrm{Alg}^{\mathrm{op}}_{\mathrm{Lie}^\pi_k}$$

is the unique sifted-colimit-preserving extension of the above functor $\mathrm{Poly}^{\mathrm{aug}}_k \to \mathrm{Alg}^{\mathrm{op}}_{\mathrm{Lie}^\pi_k}$.

We have thereby completed our goal of generalising Koszul duality from associative algebras to commutative algebras.

Next week, we will discuss explicit models for partition Lie algebras, and in particular show that in characteristic zero, they are equivalent to differential graded Lie algebras. Moreover, will see that partition Lie algebras control derived deformation functors, and introduce Quillen's cotangent complex formalism along the way.

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