∞-Categories and Deformation Theory

Lecture 6. Towards formal Moduli Problems

In this lecture, we will tie up a few loose ends from last week and then begin discussing the relationship between Lie algebras and deformation functors.

5.1. **Koszul duality for complete local Noetherian algebras.** Let k be a field. In the previous lecture, we set up the Koszul duality functor

$$\mathfrak{D} : \mathrm{SCR}_k^{\mathrm{aug}} \longrightarrow \mathrm{Alg}^{\mathrm{op}}_{\mathrm{Lie}_k^\pi}, \ R \mapsto \cot(R)^\vee$$

from augmented simplicial commutative k-algebras to *partition Lie algebras*; we introduced these new objects in Definition 5.27 as algebras over a monad Lie_k^π on the ∞-category of chain complexes.

For suitably nice augmented animated k-algebras $R \in \mathrm{SCR}_k^{\mathrm{aug}}$, the complex $\cot(R)^\vee$ with its partition Lie algebra structure remembers the structure of R.

Theorem 5.1 ([BM19]). *The functor* $\mathfrak{D} : \mathrm{SCR}_k^{\mathrm{aug}} \longrightarrow \mathrm{Alg}^{\mathrm{op}}_{\mathrm{Lie}_k^\pi}, \ R \mapsto \cot(R)^\vee$ *last week restricts to a contravariant equivalence between*

(1) the full subcategory

$$\mathrm{SCR}_k^{\mathrm{cN}} \subset \mathrm{SCR}_k^{\mathrm{aug}}$$

spanned by all R for which $\pi_0(R)$ is a complete local Noetherian ring and $\pi_i(R)$ is a finitely generated $\pi_0(R)$-module for all i.

(2) the (opposite of the) full subcategory

$$\mathrm{Alg}_{\mathrm{Lie}_k^\pi}(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}) \subset \mathrm{Alg}_{\mathrm{Lie}_k^\pi}$$

spanned by all partition Lie algebras $\mathfrak{g}$ whose underlying chain complex is coconnective and satisfies $\dim(\pi_i(\mathfrak{g})) < \infty$ for all i.

Next week, we will give geometric models for *all* partition Lie algebras.

5.2. **Explicit models for Lie algebras.** We constructed the monad Lie_k^π making substantial use of the theory of ∞-categories. First, we observed that the (contravariant) tangent fibre functor

$$A \mapsto \cot(A)^\vee = (k \otimes_A L_{A/k})^\vee$$

from augmented simplicial commutative k-algebras to chain complexes over k is part of an adjunction. While the associated monad T^{naive} on Mod_k behaved badly, we could approximate it by a monad Lie_k^π which preserves filtered colimits and geometric realisations.

If $V^\bullet$ is a cosimplicial k-module with associated chain complex $\mathrm{Tot}(V^\bullet)$, then

$$\mathrm{Lie}_k^\pi(\mathrm{Tot}(V^\bullet)) \simeq \bigoplus_n \mathrm{Tot}\left(\widetilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, k) \otimes (V^\bullet)^{\otimes n}\right)^{\Sigma_n}.$$

Here $\Sigma|\Pi_n|^\diamond$ is a simplicial Σ_n-complex known as the n^{th} (doubly suspended) *partition complex.* For $d > 0$, the nondegenerate d-simplices of $\Sigma|\Pi_n|^\diamond$ correspond to chains of increasingly coarse partitions

$$[\hat{0} = x_0 < x_1 < \ldots < x_t = \hat{1}]$$

of the set $\{1, \ldots n\}$.

One can also construct an explicit model category for the ∞-category of partition Lie algebras.

Partition Lie algebras in characteristic zero. When our ground field k is of characteristic zero, partition Lie algebras recover a familiar notion:

Definition 5.2 (Differential graded Lie algebras)**.** Let k be a field of characteristic zero. A *differential graded Lie algebra* ('DGLA') over k is a complex

$$\ldots \to \mathfrak{g}_2 \to \mathfrak{g}_1 \to \mathfrak{g}_0 \to \mathfrak{g}_{-1} \to \mathfrak{g}_{-2} \to \ldots$$

with a bilinear map $[-,-] : \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j}$ satisfying the following rules:

(Antisymmetry) $[x,y] = (-1)^{|x||y|+1}[y,x]$

(Jacobi identity) $(-1)^{|x||z|}[[x,y],z] + (-1)^{|z||y|}[[z,x],y] + (-1)^{|y||x|}[[y,z],x] = 0$

(Leibniz rule) $d([x,y]) = [dx,y] + (-1)^{|x|}[x,dy]$.

Notation 5.3. The category $\mathbf{dgla}_k$ of differential graded Lie algebras admits the structure of a left proper combinatorial model category (c.f. e.g. [Lur11, Proposition 2.1.10]) whose weak equivalences are the quasi-isomorphisms and whose fibrations are the levelwise surjections. We write dgla_k for the underlying ∞-category of $\mathbf{dgla}_k$.

Recall from Exercise 5.13 that if k is a field of characteristic 0, the category $\mathbf{cdga}_k$ of commutative differential graded k algebras carries a model structure whose weak equivalences are the quasi-isomorphisms and whose fibrations are the levelwise surjections. Write cdga_k for the underlying ∞-category of $\mathbf{cdga}_k$. We will also need the model category $\mathbf{cdga}_k^{\mathrm{aug}} = (\mathbf{cdga}_k)_{/k}$ of augmented commutative differential graded k-algebras and its underlying ∞-category $\mathrm{cdga}_k^{\mathrm{aug}}$.

To compare differential graded Lie algebras with partition Lie algebras, we will rely on the following well-known construction:

Construction 1 (Chevalley–Eilenberg complex)**.** *Given a differential graded Lie algebra $\mathfrak{g} \in \mathbf{dgla}_k$, consider its (homological) Chevalley–Eilenberg complex*

$$\mathrm{CE}_*(\mathfrak{g}) = (\mathrm{Sym}^*(\mathfrak{g}[1]), D).$$

Here $\mathrm{Sym}^*(\mathfrak{g}[1])$ is the sum of all symmetric powers of the underlying graded vector space of $\mathfrak{g}[1]$. The differential D sends the product of homogeneous elements x_i in degree p_i to

$$D(x_1 \ldots x_n) = \sum_{1 \le i \le n} (-1)^{p_1+\ldots+p_{i-1}} x_1 \ldots x_{i-1} dx_i x_{i+1} \ldots x_n$$
$$+ \sum_{1 \le i < j \le n} (-1)^{p_i(p_{i+1}+\ldots+p_{j-1})} x_1 \ldots x_{i-1} x_{i+1} \ldots x_{j-1} [x_i, x_j] x_{j+1} \ldots x_n.$$

Write $\mathrm{CE}^*(\mathfrak{g})$ for the linear dual of $\mathrm{CE}_*(\mathfrak{g})$, and define a graded-commutative multiplication on $\mathrm{CE}^*(\mathfrak{g})$ by declaring the product of $f \in \mathrm{CE}^p(\mathfrak{g})$ and $g \in \mathrm{CE}^q(\mathfrak{g})$ to be the element $fg \in \mathrm{CE}^{n+m}(\mathfrak{g})$ satisfying

$$(fg)(x_1 \ldots x_n) = \sum_{S,T} \epsilon(S,T) f(x_{i_1} \ldots x_{i_m}) g(x_{j_1} \ldots x_{j_{n-m}}).$$

Here $x_i \in \mathfrak{g}_{r_i}$ are homogeneous elements, the sum is indexed by disjoint sets $S = \{i_1, \ldots, i_m\}$, $T = \{j_1, \ldots, j_{n-m}\}$ with $S \cup T = \{1, \ldots, n\}$ and $r_{i_1} + \ldots + r_{r_m} = p$, and the sign $\epsilon(S,T)$ is given by

$$\epsilon(S,T) = \prod_{i \in S, j \in T, i<j} (-1)^{r_i r_j}.$$

Exercise 5.4. *Let* $\mathrm{U}\mathfrak{g}$ *be the universal enveloping algebra of* $\mathfrak{g}$*. Show that there are weak equivalences*

$$\mathrm{CE}_*(\mathfrak{g}) \simeq k \otimes^L_{\mathrm{U}\mathfrak{g}} k \qquad \mathrm{CE}^*(\mathfrak{g}) \simeq \mathbb{R}\mathrm{Hom}_{\mathrm{U}\mathfrak{g}}(k,k).$$

Noting that $\mathrm{CE}^*(\mathfrak{g})$ *is naturally augmented, the above construction defines a functor*

$$\mathrm{CE}^* : \mathbf{dlga}_k^{\mathrm{op}} \to \mathbf{cdga}_k^{\mathrm{aug}}.$$

Definition 5.5. The *Chevalley-Eilenberg cochains functor*

$$\mathrm{CE}^{\mathrm{dg}} : \mathrm{dgla}_k^{\mathrm{op}} \to \mathrm{cdga}_k^{\mathrm{aug}}$$

from the ∞-category of differential graded Lie algebras to the ∞-category of augmented commutative differential graded k-algebras is obtained from CE^* by inverting weak equivalences.

This functor preserves limits and therefore admits a left adjoint

$$\mathfrak{D}^{\mathrm{dg}} : \mathrm{cdga}_k^{\mathrm{aug}} \to \mathrm{dgla}_k^{\mathrm{op}}$$

Warning 5.6. *The functor* CE^* *does not admit a left adjoint as a functor of* 1*-categories, and to describe* $\mathfrak{D}^{\mathrm{dg}}$ *explicitly, it is more convenient to work with* L_∞*-algebras rather than d.g. Lie algebras.*

The above ingredients allow us to prove:

Proposition 5.7. *Let* k *be a field of characteristic zero. The composite*

$$\mathrm{Alg}_{\mathrm{Lie}_k^\pi} \to \mathrm{Mod}_k \xrightarrow{\Sigma^{-1}} \mathrm{Mod}_k$$

of the forgetful functor and the shift functor lifts to a canonical equivalence

$$\mathrm{Alg}_{\mathrm{Lie}_k^\pi} \xrightarrow{\simeq} \mathrm{dgla}_k$$

along the forgetful functor $\mathrm{dgla}_k \to \mathrm{Mod}_k$*.*

Proof. We consider the following pair of adjunctions:

$$\mathrm{cdga}_k^{\mathrm{aug}} \underset{\mathrm{CE}^{\mathrm{dg}}}{\overset{\mathfrak{D}^{\mathrm{dg}}}{\rightleftarrows}} \mathrm{dgla}_k^{\mathrm{op}} \underset{\mathrm{free}_{\mathrm{dgla}}}{\overset{\mathrm{forget}_{\mathrm{dgla}}}{\rightleftarrows}} \mathrm{Mod}_k^{\mathrm{op}}.$$

By a straightforward computation (cf. [Lur11, Proposition 2.2.15]), we have

$$\mathrm{CE}^{\mathrm{dg}}(\mathrm{free}_{\mathrm{dgla}}(V)) \simeq k \oplus \Sigma^{-1}V^\vee = \mathrm{sqz}_k(\Sigma^{-1}V^\vee).$$

Taking adjoints, we obtain an equivalence

$$\mathrm{forget}_{\mathrm{dgla}}(\mathfrak{D}^{\mathrm{dg}}(A)) \simeq \Sigma^{-1}\operatorname{cot}^{\mathrm{aug}}(A)^\vee.$$

Hence the composite of the above adjunctions is equivalent to

$$\mathrm{cdga}_k^{\mathrm{aug}} \underset{\mathrm{sqz}((\Sigma-)^\vee)}{\overset{\Sigma^{-1}\operatorname{cot}^{\mathrm{aug}}(-)^\vee}{\rightleftarrows}} \mathrm{Mod}_k^{\mathrm{op}}$$

Inserting the unit $\mathrm{id} \to \mathfrak{D}^{\mathrm{dg}} \circ \mathrm{CE}^{\mathrm{dg}}$, we obtain a map of monads

$$\mathrm{Lie}_k^{\mathrm{dg}}(-) = \mathrm{forget}_{\mathrm{dgla}} \circ \mathrm{free}_{\mathrm{dgla}}(-) \longrightarrow \Sigma^{-1}(\operatorname{cot}^{\mathrm{aug}}(\mathrm{sqz}^{\mathrm{aug}}(\Sigma-)^\vee)^\vee) \simeq \Sigma^{-1}T^{\mathrm{naive}}\Sigma(-)$$

Observe that $\mathrm{Lie}_k^{\mathrm{dg}}$ preserves the full subcategory $\mathrm{Mod}_{k,\le -1}^{\mathrm{ft}} \subset \mathrm{Mod}_k$ of (-1)-coconnective chain complexes V with $\dim(\pi_i(V)) < \infty$ for all i, as Lie brackets decrease degree.

By (a variant of) [Lur07, Lemma 2.3.5], the unit map $\mathfrak{g} \to \mathfrak{D}^{\mathrm{dg}}(\mathrm{CE}^{\mathrm{dg}}(\mathfrak{g}))$ is an equivalence for all differential graded Lie algebras $\mathfrak{g}$ with underlying chain complex in $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq -1}$. Hence the above transformation of monads

$$\mathrm{Lie}^{\mathrm{dg}}_k(-) \to \Sigma^{-1} T^{\mathrm{naive}} \Sigma(-)$$

restricts to an equivalence on $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq -1}$. We obtain an equivalence of monads

$$(\Sigma \, \mathrm{Lie}^{\mathrm{dg}}_k \, \Sigma^{-1})|_{\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}} \simeq \mathrm{Lie}^{\pi}_k |_{\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}}$$

As $\Sigma \, \mathrm{Lie}^{\mathrm{dg}}_k \, \Sigma^{-1}$ preserves sifted colimits, Construction 5.26(3) gives an equivalence of monads $\Sigma \, \mathrm{Lie}^{\mathrm{dg}}_k \, \Sigma^{-1} \simeq \mathrm{Lie}^{\pi}_k$, which implies the claim. □

Partition Lie algebras in characteristic p. Partition Lie algebras over a field k of characteristic p are richer structures. They can be modelled by *simplicial-cosimplicial restricted* $\mathbf{Lie}^{\pi}_k$*-algebras*, that is, by cosimplicial-simplicial k-modules equipped with additional operations.

The following objects parametrise the barycentric subdivision of the partition complex:

Definition 5.8. A *nested chain of partitions* of $\{1, \ldots, n\}$ is a pair

$$(\sigma, S)$$

where

$$\sigma = [\hat{0} = x_0 < x_1 < \ldots < x_t = \hat{1}]$$

is a chain of increasingly coarse partitions of $\{1, \ldots, n\}$ and

$$S = (S_0 \subseteq \ldots \subseteq S_d)$$

is a chain of increasing subsets of $S_d = \{0, \ldots, t\}$.

Construction 5.9. Let k be a field. A simplicial-cosimplicial restricted $\mathbf{Lie}^{\pi}_k$-algebra is a simplicial object in cosimplicial restricted $\mathbf{Lie}^{\pi}_k$-algebras. To equip a cosimplicial k-module

$$\mathfrak{g}^0 \rightrightarrows \mathfrak{g}^1 \ \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \ \mathfrak{g}^2 \ \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \ \cdots$$

with the structure of a *cosimplicial restricted* $\mathbf{Lie}^{\pi}_k$*-algebra*, we must specify an element

$$\{a_1, \ldots, a_r\}_{(\sigma,S)} \in \mathfrak{g}^d$$

for any nested chain

$$(\sigma, S) = \big([\hat{0} = x_0 < \cdots < x_t = \hat{1}], S_0 \subseteq \cdots \subseteq S_d\big) \in \mathbf{Lie}^{\pi}_k(r)^d$$

and any tuple

$$\mathbf{a} = (a_1, \ldots, a_r) \in \mathfrak{g}^d.$$

Moreover, for any tuple $\mathbf{a} = (a_1, \ldots, a_r)$ in $\mathfrak{g}^d$ and any nested chain $(\sigma, S) \in \mathbf{Lie}^{\pi}_k(r)^d$, we must specify a 'divided power' element

$$\gamma_{(\sigma,S)}(a_1, \ldots, a_r) \in \mathfrak{g}^d.$$

These elements are then required to satisfy various relations specified in [BCN21, Construction 5.43].

In [BCN21, Theorem 5.42], it is shown that inverting weak equivalences of simplicial-cosimplicial restricted $\mathbf{Lie}^{\pi}_k$-algebras gives rise to the ∞-category of partition Lie algebras.

5.3. **The cotangent complex.** Before proceeding to moduli problems, we will introduce one more important piece of technology: the *cotangent complex* formalism of André and Quillen. It is a derived variant of Kähler differentials, and generalises the cotangent fibre discussed in the last lecture.

Kähler differentials. Let us begin by fixing a map of ordinary commutative rings

$$A \to C,$$

together with a C-module M.

Definition 5.10. The *trivial square-zero extension*

$$\mathrm{sqz}_C(M)$$

of C *by* M is the commutative C-algebra with underlying abelian group $C \oplus M$ and multiplication

$$(c_1, m_1) \cdot (c_2, m_2) := (c_1 c_2, c_1 m_2 + c_2 m_1).$$

Exercise 5.11. *Show that the space of A-algebra sections of the projection $C \oplus M \to C$ is isomorphic to the C-module* $\mathrm{Der}_A(C, M)$ *of A-linear derivations $D : C \to M$.*

Holding the morphism $A \to C$ fixed and varying M, we obtain a functor $M \mapsto \mathrm{Der}_A(C, M)$, which is representable by the module of Kähler differentials

$$\Omega^1_{C/A},$$

i.e. the free C-module generated by symbols $\{dc\}_{c \in C}$ modulo the relations

$$da = 0 \qquad d(c_1 + c_2) = dc_1 + dc_2 \qquad d(c_1 c_2) = c_1 dc_2 + c_2 dc_1 \qquad , a \in A, c_1, c_2 \in C.$$

We obtain an adjunction

$$F : \{A\text{-algebras}/C\} \leftrightarrows \{C-\text{modules}\} : G$$

$$B \mapsto C \otimes_B \Omega_{B/A}$$

$$\mathrm{sqz}_C(M) \leftarrow M$$

The cotangent complex. To define the cotangent complex of $A \to C$, let us restrict the functor $F : B \mapsto C \otimes_B \Omega^1_{B/A}$ to the full subcategory $\mathcal{C} \subset \mathrm{CR}_{A//C}$ of A-algebras $A[x_1, \ldots, x_n]$ with a map to C. Let us moreover think of F as taking values in the ∞-category of $\mathrm{Mod}_{C,\geq 0}$ of connective chain complexes over C.

Taking the nonabelian left derived functor in the sense of Definition 5.6, we obtain a functor

$$\mathbb{L}F : \mathrm{CAlg}^{\mathrm{an}}_{A//C} \to \mathrm{Mod}_{C,\geq 0}.$$

Definition 5.12. The *(relative) cotangent complex* of $A \to C$ is given by $L_{C/A} := (\mathbb{L}F)(C)$.

To compute $L_{C/A}$ in examples, we we can first pick a weakly equivalent cofibrant simplicial A-algebra $P_\bullet$ over C, then apply the functor F levelwise to obtain the simplicial abelian group

$$C \otimes_{P_\bullet} \Omega_{P_\bullet/A},$$

and finally apply Dold-Kan to obtain a connective chain complex of C-modules.

Exercise 5.13.

a) *Use extension-of-scalars functors to construct the category* $\mathrm{PolyMod}^{\mathrm{ff}}$ *of pairs (A, M) with A a polynomial ring and M a finite free A-module.*
b) *Use* $\mathrm{PolyMod}^{\mathrm{ff}}$ *to define the cotangent complex $L_{C/A}$ for maps of animated rings $A \to C$.*

The cotangent complex is very computable in practice. We state some of its most important properties without proof:

Proposition 5.14.

(1) *If $C = A[x_1, \dots, x_n]$ is a polynomial ring, then $L_{C/A}$ is given by $\Omega^1_{C/A}$ concentrated in degree 0.*
(2) *For any morphisms $A \to B \to C$, there is a cofibre sequence in $\mathrm{Mod}_{C,\geq 0}$ of the form*

$$C \otimes_B L_{B/A} \to L_{C/A} \to L_{C/B}.$$

(3) *Given morphisms of animated rings $A \to B$ and $A \to C$, we have a canonical equivalence*

$$C \otimes_A L_{B/A} \xrightarrow{\simeq} L_{B \otimes_A C/C}.$$

(4) *If $A \to B$ is surjective with kernel I generated by a regular sequence, then $L_{B/A}$ is given by I/I^2 concentrated in homological degree 1.*

Exercise 5.15.

(1) *Show that if $C = A[x_1, \dots x_n]/(f_1, \dots f_m)$ with $f_1, \dots f_m$ a regular sequence, then*

$$L_{C/A} \simeq (\dots \to 0 \to 0 \to I/I^2 \to C \otimes_{A[x_1,\dots x_n]} \Omega_{A[x_1,\dots x_n]/A}).$$

(2) *Identify the nonzero differential appearing in the complex $L_{C/A}$ above.*

The cotangent complex generalises the cotangent fibre from Definition 5.18:

Exercise 5.16. *Show that for A an animated ring and $R \in \mathrm{CAlg}_A^{\mathrm{an,aug}}$ an augmented animated A-algebra, there is an equivalence*

$$\cot(R) \simeq A \otimes_R L_{R/A}.$$

Hint: first prove the case where R is a polynomial A-algebra.

We are now finally ready to return to deformation functors and discuss their relation with Lie algebras.

5.4. **Classical deformation functors from differential graded Lie algebras.** To begin with, let us discuss classical deformation functors over a field k of characteristic 0.

Following Schlessinger [Sch68], we define:

Definition 5.17 (Deformation functor)**.** A *deformation functor* is a functor

$$F : \mathrm{CR}_k^{\mathrm{art}} \to \mathrm{Set}$$

from the category of Artinian local commutative k-algebras with residue field k to the category of sets such that

(1) $F(k)$ has one element;
(2) for any pullback square

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ A & \longrightarrow & A'' \end{array}$$

with $A' \to A''$ surjective, the induced map $\alpha : F(A) \to F(A') \times_{F(A'')} F(A)$ is surjective;
(3) the map α is bijective whenever the lower horizontal arrow in the above square is given by

$$k[\epsilon]/\epsilon^2 \to k.$$

Many deformation functors in characteristic 0 are controlled by differential graded Lie algebras. To make this idea precise, we define:

Definition 5.18 (Deformation functor associated to a differential graded Lie algebra)**.**
Given a differential graded Lie algebra $\mathfrak{g}$, the classical deformation functor

$$D_{\mathfrak{g}} : \mathrm{CR}_k^{\mathrm{art}} \to \mathrm{Set}$$

associated to $\mathfrak{g}$ sends a given local Artinian $A \in \mathrm{CR}_k^{\mathrm{art}}$ with maximal ideal $\mathfrak{m}_A$ (and residue field k) to the set

$$D_{\mathfrak{g}}(A) = \left\{ x \in \mathfrak{g}_{-1} \otimes \mathfrak{m}_A \mid dx + \frac{1}{2}[x,x] = 0 \right\} \Big/ \simeq \ ,$$

where the equivalence relation $\simeq$ identifies each x with

$$x + \sum_{n=0}^{\infty} \frac{[a,-]^{\circ n}}{(n+1)!}([a,x] - da)$$

where a ranges over $\mathfrak{g}_0 \otimes \mathfrak{m}_A$.

Here we have equipped $\mathfrak{g} \otimes \mathfrak{m}_A$ with the structure of a differential graded Lie algebra by setting

$$[x \otimes a, y \otimes b] := (-)^{\deg(a)\deg(y)}[x,y] \otimes ab.$$

Exercise 5.19. *Check that $D_{\mathfrak{g}}$ defines a deformation functor in the sense of Definition 5.17.*

We will now discuss a simple example of a deformation functor controlled by a differential graded Lie algebra, following [M09].

Example 5.20 (Chain complexes)**.** Let us fix a perfect complex

$$M = (\ldots \to 0 \to M_n \xrightarrow{\partial} M_{n-1} \xrightarrow{\partial} M_{n-2} \xrightarrow{\partial} \ldots \xrightarrow{\partial} M_0 \to 0 \to \ldots)$$

over a field k of characteristic 0. Given $A \in \mathrm{CR}_k^{\mathrm{art}}$ with maximal ideal $\mathfrak{m}_A$, a deformation of M over A is a perfect complex

$$\widetilde{M} = (\ldots 0 \to A \otimes_k M_n \xrightarrow{\widetilde{\partial}} A \otimes_k M_{n-1} \xrightarrow{\widetilde{\partial}} A \otimes_k M_{n-2} \xrightarrow{\widetilde{\partial}} \ldots \xrightarrow{\widetilde{\partial}} A \otimes_k M_0 \to 0 \to \ldots)$$

together with an isomorphism $\widetilde{M} \otimes_A k \cong M$.

Using the decomposition

$$A \otimes_k M_i \ \cong\ M_i \oplus (\mathfrak{m}_A \otimes_k M_i), \qquad (1)$$

we can write the differential $\widetilde{\partial}$ of $\widetilde{M}$ as (the A-linear extension of)

$$\partial + \phi,$$

where $\phi \in \mathrm{Hom}^1(M,M) \otimes \mathfrak{m}_A$. The condition $\widetilde{\partial}^2 = 0$ is then equivalent to

$$\partial\phi + \phi\partial + \phi^2 = 0.$$

We now consider the differential graded Lie algebra

$$(\mathfrak{g}_M)_* = \mathrm{Hom}^{-*}(M,M)$$

with differential

$$d(f) = \partial f - (-1)^{\deg f} f \partial$$

and Lie bracket

$$[f,g] = fg - (-1)^{\deg(f)\deg(g)} gf.$$

Then elements $\phi \in \mathrm{Hom}^1(M,M) \otimes \mathfrak{m}_A$ satisfying $\partial\phi + \phi\partial + \phi^2 = 0$ are exactly elements

$$\phi \in (\mathfrak{g}_M)_{-1} \otimes \mathfrak{m}_A$$

satisfying

$$d\phi + \frac{1}{2}[\phi,\phi].$$

Two elements $\phi_1, \phi_2 \in (\mathfrak{g}_M)_{-1} \otimes \mathfrak{m}_A = \mathrm{Hom}^1(M,M) \otimes \mathfrak{m}_A$ define isomorphic deformations if there is a diagram

$$\begin{array}{ccccccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & A \otimes_k M_n & \xrightarrow{\partial+\phi_1} & A \otimes_k M_{n-1} & \xrightarrow{\partial+\phi_1} & \dots & \xrightarrow{\partial+\phi_1} & A \otimes_k M_0 & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \\
\dots & \longrightarrow & 0 & \longrightarrow & A \otimes_k M_n & \xrightarrow{\partial+\phi_2} & A \otimes_k M_{n-1} & \xrightarrow{\partial+\phi_2} & \dots & \xrightarrow{\partial+\phi_2} & A \otimes_k M_0 & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

such that α reduces to the identity mod $\mathfrak{m}_A$.

Using the decomposition (1), we can write α as (the A-linear extension of)

$$\mathrm{id} + \eta$$

for some $\eta \in \mathrm{Hom}^0(M,M) \otimes \mathfrak{m}_A$.

Exercise 5.21.

(1) *Show that* $\mathrm{id} + \eta$ *can be written as*

$$\mathrm{id} + \eta = e^a$$

for a suitably chosen $a \in \mathrm{Hom}^0(M,M) \otimes \mathfrak{m}_A$*;*

(2) *use the equation* $(\partial + \phi_2) = e^a \circ (\partial + \phi_1) \circ e^{-a}$ *to prove that*

$$\phi_2 = \phi_1 + \sum_{n=0}^{\infty} \frac{[a,-]^{\circ n}}{(n+1)!}([a,\phi_1] - da).$$

In the next lecture, we will see other examples of classical deformation functors controlled by differential graded Lie algebras, including the deformation functor of a smooth and proper variety discussed in the first lecture.

We will then see why classical deformation functors are not sufficient for some problems in deformation theory, and introduce formal moduli problems to resolve this obstacle.

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