Lecture 7. Classical deformation functors from Lie algebras

Last week, we constructed a deformation functor $D_{\mathfrak{g}}$ for every differential graded Lie algebra $\mathfrak{g}$ in characteristic 0. Moreover, we saw that the deformation functor of a perfect chain complex M is controlled by the differential graded Lie algebra $\mathfrak{g}_M = \mathrm{Hom}^{-*}(M,M)$.

Before exploring *derived* deformation functors in general characteristics in week 8 and 9, we will explain a second method of constructing classical deformation functors in characteristic zero via *semicosimplicial Lie algebras* (following the treatment in Manetti's book [Man22], which we recommend). This method sometimes facilitates a more algebraic treatment of deformation functors – we will illustrate this for vector bundles and complex varieties.

7.1. From nilpotent Lie algebras to groups.

But first, let us recall a classical piece of algebra. Fix a field k of characteristic zero, and let R be an associative k-algebra containing a nilpotent ideal I. Consider the exponential function

$$e : I \to 1 + I$$
$$x \mapsto e^x = \sum_{n\geq 0} \frac{x^n}{n!}$$

The formula $e^x e^y = e^{x+y}$ need not hold in the non-commutative setting, but there is always a product

$$\bullet : I \times I \to I$$

such that the following equation holds true for all $x, y \in I$:

$$e^x e^y = e^{x\bullet y}.$$

To define this *Baker–Campbell–Hausdorff* product $(x,y) \mapsto x \bullet y$, we only need the underlying nilpotent Lie algebra structure on I, which is defined by commutators

$$[x,y] = xy - yx.$$

The first few terms are given by

$$x \bullet y = x + y + \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] + \frac{1}{12}[y,[y,x]]) + \frac{1}{24}[x,[y,[y,x]]] + \cdots.$$

The full formula can be made explicit as a sum

$$x \bullet y = \sum_{n\geq 0} Z_n(x,y)$$

where $Z_n(x,y)$ are defined in terms of iterated Lie brackets and the *Bernoulli numbers* $B_0, B_1, B_2, \dots$ appearing in the series expansion

$$\frac{x}{e^x - 1} = \sum_n \frac{B_n}{n!} x^n.$$

We refer to [Man22, Section 2.5] for further details.

Definition 7.1 (From nilpotent Lie algebras to groups)**.** Given a nilpotent Lie algebra $\mathfrak{g}$, we define its exponential group $\exp(\mathfrak{g})$ as the set of expressions $\{e^x \mid x \in \mathfrak{g}\}$ with the product

$$e^x e^y = e^{x\bullet y}.$$

Exercise 7.2. *Familiarise yourself with the definition of the Baker–Campbell–Hausdorff product and verify that* $\exp(\mathfrak{g})$ *defines a group.*

7.2. **From semicosimplicial Lie algebras to deformation functors.** Semicosimplicial Lie algberas are functors from the category Δ^{inj} of finite linearly ordered sets and injections to the category of Lie algebras over k, i.e. diagrams of the form

$$\mathfrak{g}^\bullet = \left(\mathfrak{g}^0 \rightrightarrows \mathfrak{g}^1 \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots \right).$$

Definition 7.3 (Deformation functor associated to a semicosimplicial Lie algebra)**.** Given a semicosimplicial Lie algebra $\mathfrak{g}^\bullet$, the *deformation functpr $D_{\mathfrak{g}^\bullet}$ associated to* $\mathfrak{g}^\bullet$ is defined as

$$D_{\mathfrak{g}} : \mathrm{CR}_k^{\mathrm{art}} \to \mathrm{Set}, \qquad A \quad \mapsto \quad \left\{ e^x \in \exp(\mathfrak{m}_A \otimes \mathfrak{g}^1) \mid e^{\delta_1(x)} = e^{\delta_2(x)} e^{\delta_0(x)} \right\} \Big/ \exp(\mathfrak{m}_A \otimes \mathfrak{g}^0)$$

Here the group $\exp(\mathfrak{m}_A \otimes \mathfrak{g}^0)$ acts via the rule

$$e^a * e^x := e^{\delta_1(a)} e^x e^{-\delta_0(a)}.$$

Exercise 7.4. *Verify that the above action is indeed well-defined.*

Thinking of semicosimplicial Lie algebras as Δ^{inj}-indexed diagrams in the model category of differential graded Lie algebras, we can compute their homotopy limit. The following notation will be useful:

Notation 7.5. Given $n \geq 0$, let us write

$$\Omega^n = k[x_0, \ldots, x_n, dx_0, \ldots, dx_n]/(x_0 + \ldots + x_n = 1, dx_0 + \ldots + dx_n = 0)$$

for the differential graded k-algebra of polynomial differential forms on the standard n-simplex

$$\Delta^n = \{(x_0, \ldots, x_n) \mid x_0 + \ldots + x_n = 1)\} \subset \mathbb{R}^{n+1}.$$

Definition 7.6 (From semicosimplicial to differential graded Lie algebras)**.** Given a semicosimplicial object in differential graded Lie algebras $\mathfrak{g}^\bullet$, we can define a differential graded Lie algebra

$$\mathrm{Tot}(\mathfrak{g}^\bullet) := \left\{ (x_n) \in \prod_{n \geq 0} \Omega^n \otimes \mathfrak{g}^n \mid (\delta_k^* \otimes \mathrm{id}) x_n = (\mathrm{id} \otimes \delta_k) x_{n-1} \ \forall 0 \leq k \leq n \right\},$$

where the differential and the Lie bracket are inherited from $\prod_{n \geq 0} \Omega^n \otimes \mathfrak{g}^n$.

Exercise 7.7. *Show that* $\mathrm{Tot}(\mathfrak{g}^\bullet)$ *computes the homotopy limit of* $g^\bullet$*.*

Using the work of Hinich [Hin96], Manetti (cf. [Man22, Corollary 7.6.6]) relates Definition 7.3 and Definition 6.18:

Theorem 7.8. *Given a semicosimplicial Lie algebra* $\mathfrak{g}^\bullet$*, there is an isomorphism of deformation functors*

$$D_{\mathfrak{g}^\bullet} \cong D_{\mathrm{Tot}(\mathfrak{g})}.$$

7.3. **Deformations of sheaves of $\mathcal{O}_X$-modules on complex varieties.** Let us fix a complex manifold $(X, \mathcal{O}_X)$ and a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on X.

Definition 7.9. A deformation of $\mathcal{F}$ over some local Artinian $\mathbb{C}$-algebra $A \in \mathrm{CR}^{\mathrm{art}}_{\mathbb{C}}$ is a pair

$$(\mathcal{F}_A, \alpha),$$

where $\mathcal{F}_A$ is a sheaf of $A \otimes \mathcal{O}_X$-modules that is flat over A and α is an isomorphism $\alpha : \mathbb{C} \otimes_A \mathcal{F}_A \cong \mathcal{F}$.

An isomorphism of deformations

$$(\mathcal{F}_A, \alpha) \to (\mathcal{F}'_A, \alpha')$$

is an $A \otimes \mathcal{O}_X$-linear isomorphism of sheaves $f : \mathcal{F}_A \to \mathcal{F}'_A$ that $\alpha' \circ (\mathbb{C} \otimes_A f) = \alpha$.

The deformation functor

$$\mathrm{Def}_{\mathcal{F}}$$

of $\mathcal{F}$ sends $A \in \mathrm{CR}^{\mathrm{art}}_{\mathbb{C}}$ to the set of deformations of $\mathcal{F}$ to A up to isomorphism.

The first step in finding a Lie algebra governing $\mathrm{Def}_{\mathcal{F}}$ is to find a Lie algebra whose exponential is the group of automorphisms of the trivial deformation. To this end, note that the group

$$\mathfrak{g} \coloneqq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$$

of endomorphisms of $\mathcal{F}$ carries a Lie bracket given by the commutator. We observe:

Exercise 7.10. *Given a local Artinian $\mathbb{C}$-algebra $A \in \mathrm{CR}^{\mathrm{art}}_{\mathbb{C}}$, show that the Lie algebra*

$$\mathfrak{m}_A \otimes \mathfrak{g}$$

is isomorphic to

$$\{\ g \in \mathrm{Hom}_{A \otimes \mathcal{O}_X}(A \otimes \mathcal{F}, A \otimes \mathcal{F}) \mid \mathrm{im}(g) \subset \mathfrak{m}_A \otimes \mathcal{F}\ \}$$

with its commutator bracket.

Proceeding as in Definition 7.1, we obtain a group

$$\exp(\mathfrak{m}_A \otimes \mathfrak{g}) = \exp(\{g : \mathrm{Hom}_{A \otimes \mathcal{O}_X}(A \otimes \mathcal{F}, A \otimes \mathcal{F}) \mid \mathrm{im}(g) \subset \mathfrak{m}_A \otimes \mathcal{F}\}).$$

The exponential

$$e^g \quad \mapsto \quad \mathrm{id} + g + \frac{g^{\circ 2}}{2!} + \frac{g^{\circ 3}}{3!} + \dots$$

gives rise to a group homomorphism from $\exp(\mathfrak{m}_A \otimes \mathfrak{g})$ to the group

$$\mathrm{Aut}_{A \otimes \mathcal{O}_X}(A \otimes \mathcal{F}) = \{\ f : \mathrm{Hom}_{A \otimes \mathcal{O}_X}(A \otimes \mathcal{F}, A \otimes \mathcal{F}) \mid \mathrm{im}(f) - \mathrm{id} \subset \mathfrak{m}_A \otimes \mathcal{F}\ \}.$$

Exercise 7.11. *Prove that this map is an isomorphism.*

Pick an open cover $\mathfrak{U} = \{U_i\}$ of our manifold X. We obtain a semicosimplicial Lie algebra

$$\mathfrak{g}^\bullet_{\mathcal{F}} \coloneqq \left(\prod_i \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})(U_i) \rightrightarrows \prod_{i,j} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})(U_i \cap U_j) \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; \cdots \right).$$

Given an element θ in

$$\left\{ e^x \in \exp(\mathfrak{m}_A \otimes \mathfrak{g}^1_{\mathcal{F}}) \mid e^{\delta_1(x)} = e^{\delta_2(x)} e^{\delta_0(x)} \right\} = \exp\left(x \in \mathfrak{m}_A \otimes \mathfrak{g}^1_{\mathcal{F}} | \delta_2(x) \bullet \delta_0(x) = \delta_1(x) \right),$$

we obtain automorphisms

$$\theta_{ij} \in \mathrm{Aut}_{A \otimes \mathcal{O}_X|_{U_i \cap U_j}}(A \otimes \mathcal{F}|_{U_i \cap U_j})$$

satisfying the usual cocycle conditions.

We then obtain a sheaf $\mathcal{F}_\theta$ of $A \otimes \mathcal{O}_X$-modules on X by

$$\mathcal{F}_\theta(W) = \left\{ (s_i) \in \prod_i (A \otimes \mathcal{F})(U_i \cap W) \mid \forall i,j : \theta_{ij}(s_i) = s_j \right\}$$

This gluing construction gives rise to a natural transformation of deformation functors

$$\mathrm{Def}_{\mathfrak{g}^\bullet_{\mathcal{F}}} \to \mathrm{Def}_{\mathcal{F}}\,.$$

We then have (see [Man22, Theorem 4.2.3]):

Theorem 7.12. *Assume $\mathcal{F}$ is locally free, and that $\mathfrak{U} = \{U_i\}$ is a covering of X such that $\mathcal{F}|_{U_i}$ is trivial and $H^1(U_i, \mathcal{O}_{U_i}) = 0$ for all i. The above construction induces an isomorphism of functors*

$$\mathrm{Def}_{\mathfrak{g}^\bullet_{\mathcal{F}}} \xrightarrow{\cong} \mathrm{Def}_{\mathcal{F}}\,.$$

7.4. **Deformations of complex varieties.** We will now study the formal deformations of a complex manifold $(X, \mathcal{O}_X)$ (following [Man22, Sec.4.3]), these already made a brief appearance in the first lecture. More precisely, we will study the following deformation functor:

Definition 7.13 (Deformation functor of a complex manifold)**.** Given a complex manifold $(X, \mathcal{O}_X)$ and a local Artinian $\mathbb{C}$-algebra $A \in \mathrm{CR}^{\mathrm{art}}_{\mathbb{C}}$, a deformation of X to A is a pair

$$(\mathcal{O}_A, \alpha),$$

where $\mathcal{O}_A$ is a flat sheaf of A-algebras on X and $\alpha : \mathbb{C} \otimes_A \mathcal{O}_A \xrightarrow{\cong} \mathcal{O}$ is an isomorphism.

An isomorphism

$$(\mathcal{O}_A, \alpha) \to (\mathcal{O}'_A, \alpha')$$

between two deformations is a map of sheaves of A-algebras $f : \mathcal{O}_A \to \mathcal{O}'_A$ such that $\alpha' \circ (\mathbb{C} \otimes_A f) = \alpha$.

Let us write

$$\mathrm{Def}^{\heartsuit}_X(A)$$

for the set of deformations of X to A up to isomorphism.

Exercise 7.14. *Give an interpretation of X in terms of pullbacks of schemes.*

As before, the first step in finding a Lie algebra governing $\mathrm{Def}^{\heartsuit}_X$ is to find a Lie algebra whose exponential is the group of automorphisms of the trivial deformation.

But first, let us make some observations in the affine case. Fix a morphism of $\mathbb{C}$-algebras

$$S \to R.$$

Notation 7.15. Given a local Artinian $\mathbb{C}$-algebra $A \in \mathrm{CR}^{\mathrm{art}}_{\mathbb{C}}$ with maximal ideal $\mathfrak{m}_A$, write

$$\mathrm{Aut}'_{A\otimes S}(A \otimes R)$$

for the collection automorphisms of $S \otimes A$-algebras $A \otimes R \to A \otimes R$ such that the composite

$$A \otimes R \to A \otimes R \to R$$

is given by the projection.

Note that any map $\alpha : A \to B$ induces a map $\alpha : \mathrm{Aut}'_{A\otimes S}(A \otimes R) \to \mathrm{Aut}'_{B\otimes S}(B \otimes R)$.

Definition 7.16 (Extending derivations)**.** Consider the morphism

$$\overline{(-)} : \mathfrak{m}_A \otimes \mathrm{Der}_S(R, R) \to \mathrm{Der}_{S\otimes A}(A \otimes R, A \otimes R)$$

sending

$$d \otimes a\,, \qquad d \in \mathrm{Der}_S(R,R), a \in \mathfrak{m}_A$$

to the derivation

$$\overline{d \otimes a} \in \mathrm{Der}_{S \otimes A}(A \otimes R, A \otimes R), \quad (x \otimes b) \mapsto (dx \otimes ab).$$

Definition 7.17 (Exponentiating derivations)**.** Given $\phi \in \mathfrak{m}_A \otimes \mathrm{Der}_S(R, R)$, we consider the map

$$\exp(\overline{\phi}) = 1 + \overline{\phi} + \frac{\overline{\phi}^{\circ 2}}{2!} + \frac{\overline{\phi}^{\circ 3}}{3!} + \dots.$$

Exercise 7.18. *Verify that $\exp(\overline{\phi})$ is a well-defined map $A \otimes R \to A \otimes R$ and in fact belongs to $\mathrm{Aut}'_{A \otimes S}(A \otimes R)$. What is its inverse?*

Proposition 7.19. *The exponential*

$$\exp : \mathfrak{m}_A \otimes \mathrm{Der}_S(R, R) \to \mathrm{Aut}'_{A \otimes S}(A \otimes R)$$
$$\phi \mapsto \exp(\overline{\phi})$$

is bijective.

Proof. If $A = k$, then both sides just have one point and so the result is clear.

For general A, we can pick a surjection of local Artinian $\mathbb{C}$-algebras

$$f : A \to B$$

with kernel $J \subset B$ such that $\mathfrak{m}_A J = 0$ and B has shorter length. By induction, we may assume the claim holds true for B.

For injectivity, let us fix

$$\phi_1, \phi_2 \in \mathfrak{m}_A \otimes \mathrm{Der}_S(R, R)$$

with

$$\exp(\overline{\phi_1}) = \exp(\overline{\phi_2}) \in \mathrm{Aut}'_{A \otimes S}(A \otimes R).$$

As the induced morphisms in $\mathrm{Aut}'_{B \otimes S}(B \otimes R)$ agree, the injectivity of

$$\mathrm{Der}_S(R, R) \otimes \mathfrak{m}_B \to \mathrm{Aut}'_{B \otimes S}(B \otimes R)$$

implies that $\psi = \phi_2 - \phi_1$ belongs to $\mathrm{Der}_S(R, R) \otimes J$.

We now observe that

$$\overline{\phi_1}^i \overline{\psi}^j = \overline{\psi}^j \overline{\phi_1}^i = 0$$

for all $j > 0$, $i + j \geq 2$ since $\mathfrak{m}_A J = 0$, which in turn implies

$$\exp(\overline{\phi_2}) = \exp(\overline{\phi_1} + \overline{\psi}) = \exp(\overline{\phi_1}) + \overline{\psi} = \exp(\overline{\phi_2}) + \overline{\psi}.$$

Hence $\psi = 0$ and $\phi_1 = \phi_2$.

For surjectivity, pick some automorphism $\alpha \in \mathrm{Aut}'_{A \otimes S}(A \otimes R)$. By surjectivity of

$$\mathrm{Der}_S(R, R) \otimes \mathfrak{m}_B \to \mathrm{Aut}'_{B \otimes S}(B \otimes R)$$

and surjectivity of $A \to B$, we can pick $\phi \in \mathfrak{m}_A \otimes \mathrm{Der}_S(R, R)$ such that $f(\alpha) = f(\exp(\overline{\phi}))$.

But then

$$h := \alpha - \exp(\overline{\phi}) : A \otimes R \to R \otimes J$$

satisfies

$$h(ab) = h(a)\alpha(b) + \exp(\overline{\phi})(a)h(b).$$

But since J annihilated by $\mathfrak{m}_A$, we also have

$$h(a)\alpha(b) = h(a)b \qquad \exp(\overline{\phi})(a)h(b) = ah(b)$$

We conclude that $h : A \otimes R \to R \otimes J$ is a derivation and hence $\alpha = \exp(\overline{\phi}) + h = \exp(\overline{\phi} + h)$ by our earlier considerations. □

Let us return to the case of a complex manifold X with structure sheaf $\mathcal{O}_X$ and holomorphic tangent sheaf T_X. Fix an open covering $\mathcal{U} = \{U_i\}$, and consider the semicosimplicial Lie algebra

$$\mathfrak{g}_X^\bullet := \left(\prod_i T_X(U_i) \rightrightarrows \prod_{i,j} T_X(U_i \cap U_j) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \cdots \right),$$

where each Lie bracket is defined as the commutator bracket of vector fields.

Note that $T_X(U) = \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$, and Proposition 7.19 gives a natural isomorphism

$$\exp(\mathfrak{m}_A \otimes T_X(U)) \cong \mathrm{Aut}'_A(A \otimes \mathcal{O}_X(U)).$$

for every open set $U \subset X$.

Given an element θ in

$$\left\{ e^x \in \exp(\mathfrak{m}_A \otimes \mathfrak{g}_X^1) \mid e^{\delta_1(x)} = e^{\delta_2(x)} e^{\delta_0(x)} \right\} = \exp\left(x \in \mathfrak{m}_A \otimes \mathfrak{g}_X^1 \mid \delta_2(x) \bullet \delta_0(x) = \delta_1(x) \right),$$

we obtain automorphisms

$$\theta_{ij} \in \mathrm{Aut}'_A(A \otimes \mathcal{O}_X)$$

satisfying the usual cocycle conditions.

We define a sheaf $\mathcal{O}_{X,\theta}$ of A-algebras on X by

$$\mathcal{O}_{X,\theta}(W) = \{ s_i \in A \otimes \mathcal{O}_X(U_i \cap W) \mid \forall i, j : \theta_{ij}(s_i) = s_j \}$$

We then have the following result (see [Man22, Theorem 4.3.8]):

Theorem 7.20. *Assume* $\mathfrak{U} = \{U_i\}$ *is an open covering such that* $H^1(U_i, T_X) = 0$ *for all* i. *Then the assignment* $\theta \mapsto \mathcal{O}_{X,\theta}$ *induces a natural isomorphism*

$$D_{\mathfrak{g}_X^\bullet} \to \mathrm{Def}_X^\heartsuit.$$

Remark 7.21. For $\mathfrak{U}$ as above, the differential graded Lie algeba $\mathrm{Tot}(D_{\mathfrak{g}_X^\bullet})$ is quasi-isomorphic to the *Dolbeault complex*

$$\mathfrak{g}_X = \left(\mathcal{A}^{0,0}(T_X) \to \mathcal{A}^{0,1}(T_X) \to \mathcal{A}^{0,2}(T_X) \to \ldots \right),$$

where $\mathcal{A}^{0,k}(T_X)$ is locally generated by sections of the form

$$f \; d\overline{z}_{i_1} \wedge \ldots \wedge d\overline{z}_{i_k} \otimes \alpha.$$

The Lie bracket is obtained by wedging differential forms and taking the commutator of vector fields.

The Newlander–Nirenberg theorem can be used to prove directly that the differential graded Lie algebra $\mathfrak{g}_X$ controls deformations of X. We refer to [Huy05, Section 6] for a detailed treatment.

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